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ABSTRACT

This dissertation is a study on color energy and its variations in graphs. Apart from an exploration of color energy corresponding to various coloring schemes, the notion of $\mathscr{P}$-energy as a generalization of color energy has been introduced. Computation of color energy and $\mathscr{P}$-energy has been carried out using Python program, while the general results are derived using research and proof techniques in linear algebra.

The study on color energy with respect to $L(h,k)$-coloring has been examined in detail for some families of graphs such as star graph, double star, crown graph and their color complements. We have also examined the variation of color energy in the specific cases of T-coloring, radio coloring and Hamiltonian coloring for some families of graphs. Examination of color energy for improper coloring such as open neighbourhood coloring and improper C-coloring has been done. It has also been found that the color energy of a graph G is greater than the ratio of domination number times $\sqrt{2(m+m'_c)}$ to the number of vertices in G.

The concept of P-energy has been introduced as a generalization of the concept of color energy. This stems from the fact that coloring problems in essence are vertex partition problems. For any vertex partition $\mathscr{P}$ having k elements, we define the $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$ having $-1,0,1,2$ as off-diagonal entries and diagonal entries represent the cardinality of the elements in partition $\mathscr{P}$. Then $E_{\mathscr{P}}(G)$ is the sum of the absolute values of eigenvalues $\mathscr{P}$-matrix. We have obtained some properties of $\mathscr{P}$-matrix, bounds of $E_{\mathscr{P}}(G)$ for general graphs and explicit values of $E_{\mathscr{P}}(G)$ for some graph families.

Further, we obtain $\mathscr{P}$-energy of generalized Petersen graphs with different vertex partitions. Using the concept of M-coronal, which was used to determine spectra of corona of graphs, we derive $\mathscr{P}$-energy of join of graphs and their generalized complements.

In addition to this, we develop a python program to find the $\mathscr{P}$-energy of $G(p,k)$ for the vertex partitions under consideration to visualize the comparative study of $E_{\mathscr{P}}(G(p,k))$ corresponding to different partitions and examine the relation between them. Further, we present a python program to find color energy, partition energy and $\mathscr{P}$-energy of a given graph.

Key Words: Graph energy, color eigenvalues, color energy, color complement, vertex partitions, $\mathscr{P}$-matrix, $\mathscr{P}$-energy, coronal of a graph, join of graphs

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Symbols and Notations

$\Gamma_{A_{\mathscr{P}}}(\lambda)$	$\mathscr{P}$-coronal of a graph G with partition $\mathscr{P}$ and matrix $A_{\mathscr{P}}(G)$
$A_c(G)$	Color matrix of a graph G
$A_{\mathscr{P}}(G)$	$\mathscr{P}$-matrix of a graph G with respect to vertex partition $\mathscr{P}$
$E_c((\overline{G})_c)$	Color energy of complement of a graph G
$E_c(G)$	Color energy of a graph G
$E_H(G)$	Color energy of a graph G with respect to H-coloring
$E_\chi(G)$	Color energy of a graph G corresponding to proper coloring with minimum number of colors $\chi(G)$
$E_{\mathscr{P}(d)}(G)$	$\mathscr{P}$-energy of a graph G with respect to domatic partition
$E_{\mathscr{P}(d^k)}(G)$	$\mathscr{P}$-energy of a graph G with respect to k-ply domatic partition
$E_{\mathscr{P}(i)}(G)$	$\mathscr{P}$-energy of a graph G with respect to independent vertex partition
$E_{\mathscr{P}(pm)}(G)$	$\mathscr{P}$-energy of a graph G with respect to vertex partition containing perfect matching
$E_{\mathscr{P}_r}(G)$	Robust $\mathscr{P}$-energy of a graph G
$E_{\mathscr{P}_s}(G)$	Shear $\mathscr{P}$-energy of a graph G
$E_{\mathscr{P}}(\overline{G}_k)$	$\mathscr{P}$-energy of k-complement of a graph G
$E_{\mathscr{P}}(\overline{G}_{k(i)})$	$\mathscr{P}$-energy of $k(i)$-complement of a graph G
$E_{\mathscr{P}}(G)$	$\mathscr{P}$-energy of a graph G with respect to vertex partition $\mathscr{P}$
$E_{\overline{\chi}_{imp}}(G)$	Color energy of a graph G with respect to improper C-coloring
$E_{\overline{\chi}_{onc}}(G)$	Color energy of a graph G with respect to open neighborhood coloring
$E_{LC}(G)$	Color energy of a graph G with respect to $L(h,k)$-coloring
$E_{Rd}(G)$	Color energy of a graph G with respect to Radio d-coloring
$E_T(G)$	Color energy of a graph G with respect to T-coloring
$G_1 \triangledown G_2$	Join of G_1 and G_2
$E_{\mathscr{P}(td)}(G)$	$\mathscr{P}$-energy of a graph G with respect to total domatic partition

1 Introduction

Graph theory is a study of mathematical structures that are concerned with objects and their mutual relation. There are many disciplines that uses the practice of formulating mathematical models using graphs. Many other such applications of graph theory attracted researchers to study it from various perspectives. One of the famous studies in this regard is the study of graph energy.

Earlier to 1940, energy of a graph was restricted to only molecular graphs. A molecular graph is a representation of a molecular structure into a graph with certain conditions such as the degree of the graph should be less than or equals to three. These restrictions made the study of energy very narrow. Later in 1978, a renowned chemist and a mathematician Gutman [28] initiated this study for an arbitrary graph. He defined the energy of a graph as follows:

Definition 1.1. [28, 70] The energy of a graph G is the sum of absolute values of all eigenvalues of adjacency matrix of $A(G)$ where the entries are 1, if the vertices v_i and v_j are adjacent in G, and 0, if the vertices v_i and v_j are not adjacent in G.

One of the variation of graph energy is the color energy of a graph which was introduced by Adiga et al. [3] and defined it as follows:

Definition 1.2. [3] The color energy of a graph G is the sum of absolute values of the eigenvalues of color matrix of G. It can be written as

$$E_c(G) = \sum_{i=1}^{n} |\lambda_i|$$

where λ_i's are eigenvalues of the color matrix $A_c(G)$ whose entries are

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent such that } c(v_i) \neq c(v_j), \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent such that } c(v_i) = c(v_j), \\ 0 & \text{otherwise.} \end{cases}$$

where $c(v_i)$ is the color of a vertex v_i in G.

In Example 1.3, the computation of color energy of a Paw graph is described. It should be noted that the color energy of a graph G with minimum number of colors $\chi(G)$ is denoted by $E_\chi(G)$ and, color matrix and spectrum of G are denoted by $A_\chi(G)$ and $Spec_\chi(G)$ respectively [3]. $\chi(G)$ is the least number of colors needed to color G [13].

Example 1.3. Consider the paw graph as shown in Figure 1. Its chromatic number is 3.

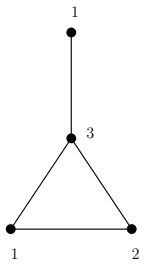


Figure 1: A paw graph G with proper vertex coloring

The color matrix of the paw graph is

$$A_\chi(G) = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

It can be verified that the eigenvalues of $A_\chi(G)$ are $-2,-1,1,2$. Therefore, $E_\chi(G) = 6$.

1.1 Basic Terminology

Throughout the dissertation, we consider undirected, simple and finite graphs of order n and size m. Unless and otherwise mentioned, for graph theoretic and spectral theoretic terminologies we refer [3, 20, 31, 37, 38, 70].

Color energy has its roots in two different areas of graph theory, namely energy of a graph and coloring of a graph. There are various types of colorings, but we mainly use the following coloring schemes.

Definition 1.4. [13] A proper vertex coloring is a function which assigns colors to the vertices of a graph such that adjacent vertices receive different color.

Definition 1.5. [13] For a graph G and a given finite set T of non-negative integers containing 0, a T -coloring of G is an assignment c of colors (positive integers) to the vertices of G such that if uw is an edge in G, then $|c(u) - c(w)| \notin T$. The T-chromatic number $\chi_T(G)$ is the minimum number of colors that can be used in a T-coloring of G.

Definition 1.6. [13] Let h,k be two non-negative integers and let G be a graph. Then $L(h,k)$-coloring of G is a function $c : \{0 \cup \mathbf{N}\} \to V(G)$ satisfying the following conditions:

1. if v_i and v_j are neighbors in G, then $|c(v_i) - c(v_j)| \geq h$,

2. if v_i and v_j are at a distance 2 in G, then $|c(v_i) - c(v_j)| \geq k$.

Definition 1.7. [13] Let G be a connected graph of order n and diameter d. For an integer k with $1 \leq k \leq d$, a radio k-coloring of G is an assignment c of colors (positive integers) to the vertices of G such that

$$d(u,v) + |c(u) - c(v)| \geq 1 + k$$

for every two distinct vertices u and v of G.

Definition 1.8. [13] For distinct vertices u and v in a connected graph G of order n, let $D(u,v)$ denote the length of a longest $u - v$ path in G. A coloring c of G is called a Hamiltonian coloring if

$$D(u,v) + |c(u) - c(v)| \geq n - 1$$

for every two distinct vertices u and v of G.

Definition 1.9. [25] The open neighborhood coloring of a graph G(V,E) is a coloring $f : V(G) \to Z^+$, such that for each $w \in V(G)$ and $\forall u, v \in N(w), f(u) \neq f(v)$.

Definition 1.10. [12] For an integer $k \geq 1$, a k-improper C-coloring of a graph $G = (V,E)$ is a coloring $\phi : V \to N$ such that for each vertex v at most k vertices in $N(v)$ receive colors different from $\phi(v)$.

Adiga et al. have introduced the concept of color complement of a graph in [3]. We will be using the following definition of color complement.

Definition 1.11. [3] If G is a graph with vertex coloring, then its color complement is a graph $(\overline{G})_c$ such that

1. u_i and u_j are adjacent in $(\overline{G})_c$ if they are non-adjacent in G with $c(u_i) \neq c(u_j)$.

2. u_i and u_j are non-adjacent in $(\overline{G})_c$ if they are adjacent in G or non-adjacent with $c(u_i) = c(u_j)$ in G.

The definitions of generalized complements by Sampathkumar et al. are as follows:

Definition 1.12. [59] Let G be a graph and $P_k = \{V_1, V_2, \ldots, V_k\}$ be its vertex partition. Then the *k-complement of* G, $(\overline{G})_k$ is obtained by removing edges between the vertices of V_i and V_j, for $i \neq j$ and adding edges between the vertices of V_i and V_j which are not in G.

Definition 1.13. [60] Let G be a graph and $P_k = \{V_1, V_2, \ldots, V_k\}$ be its vertex partition. Then the $k(i)$-*complement of* G, $(\overline{G})_{k(i)}$ is obtained by removing edges of G joining the vertices within V_r and adding the edges from complement of G between the vertices V_r.

We consider a special family of graphs that is generalized Petersen graphs conceptualized by Coxeter [18] and Watkins [68].

Definition 1.14. [68] The generalized Petersen graph $G(p,k)$ is a graph of order $n = 2p$ and $p \geq 3$ with vertex and edge set as follows: For $1 \leq k \leq p-1$ and $2k \neq n$,

1. $V[G(p,k)] = \{u_0, u_1, \ldots, u_{p-1}, v_0, v_1, \ldots, v_{p-1}\}$
2. $E[G(p,k)] = \{u_i v_i, u_i u_{i+1}, v_i v_{i+k} : i \in Z_p \text{ and } Z_p \text{ is group of integer modulo } p\}$.

Theorem 1.15. *[75] All generalized Petersen graphs are unit distance graphs*

Definition 1.16. [35] A graph that can be represented in the Euclidean plane in such a way, that vertices are represented as distinct points and edges as line segments of length one, is called a unit distance graph.

We consider the following definition of join of graphs in the present study.

Definition 1.17. [58] For two graphs G_1 and G_2, the join $G = G_1 \triangledown G_2$ is the graph obtained by joining every vertex of graph G_1 with every vertex of graph G_2.

Definition 1.18. [19] If M is a matrix associated with a graph G of order n, the M-coronal $\Gamma_M(\lambda)$ of G is the sum of elements of the matrix $(\lambda I_n - M)^{-1}$. That is, $\Gamma_M(\lambda) = \mathbf{1}_n{}^T(\lambda I_n - M)^{-1}\mathbf{1}_n$, where $\mathbf{1}_n$ is a column vector of order $n \times 1$ and $\mathbf{1}_n^T$ is its transpose and I_n is an identity matrix of order $n \times n$.

This definition is a generalization of the coronal of a graph introduced by McLeman and McNicholas for the adjacency matrix of a graph [46]. Considering the $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$ associated with the vertex partition $\mathscr{P}$ of a graph G we define the $\mathscr{P}$-*coronal of* G and denote it by $\Gamma_{A_{\mathscr{P}}}(\lambda) = \mathbf{1}_n^T(\lambda I - A_{\mathscr{P}}(G))^{-1}\mathbf{1}_n$. Our main result is an expression for the characteristic polynomial $\phi_{\mathscr{P}}(G)$ of join of graphs in terms of $\mathscr{P}$-coronal of component graphs of G.

We would be referring to the Lemma 1.19 proved by Liu and Zhang in 2019 relating $\Gamma_M(\lambda)$ with a real matrix A, the identity matrix I_n and $J_{n\times n}$, a matrix of order $n \times n$ for which each element $a_{ij} = 1$.

Lemma 1.19. *[43] Let A be an $n \times n$ real matrix. Then*

$$\det(\lambda I_n - A - \alpha J_n) = [1 - \alpha \Gamma_M(\lambda)] \det(\lambda I_n - A). \tag{1}$$

where α is a real number and λ is an eigenvalue of A.

Now, we state some of the basic results from linear algebra required for the present study:

Remark 1.20. [47] If A is a real or complex matrix of order $n \times n$ with the characteristic polynomial $\phi(G, \lambda)$ and eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

(i) A principal sub-matrix of order $r \times r$ of A is a sub-matrix consisting of the same set of r rows and r columns and a principal minor of order $r \times r$ of A is the determinant of a principal sub-matrix of order $r \times r$.

(ii) If $a_0, a_1, a_2, \ldots, a_n$ are the coefficients of $\phi(G, \lambda)$, then $(-1)^r a_r$ is the sum of principal minors of order $r \times r$.

(iii) If the r^{th} symmetric function $S_r(A)$ is the sum of the product of the eigenvalues of A taken r at a time, then it is the sum of $r \times r$ principal minors of A.

(iv) Trace of A is the sum of diagonal entries of A and it can also be represented as $tr(A) = S_1(A) = -a_1$.

(v) $\prod_{i=1}^{n} \lambda_i = |A|$.

Theorem 1.21. *[45] If λ is an eigenvalue of the matrix $(a_{ij})_{n \times n}$, then*

$$|\lambda| \leq n \max_{i,j} |a_{ij}|.$$

Lemma 1.22. *[45] The Cauchy-Schwartz inequality is stated as*

$$(\sum_{i=1}^{n} a_i b_i)^2 \leq \sum_{i=1}^{n} (a_i)^2 \sum_{i=1}^{n} (b_i)^2. \tag{2}$$

Lemma 1.23. *[66] If M, N, P, Q are matrices where M is invertible and $S = \left(\begin{smallmatrix} M & N \\ P & Q \end{smallmatrix}\right)$, then* $\det S = \det M \cdot \det[Q - PM^{-1}N]$.

Lemma 1.24. *[20] Let $A = \left(\begin{smallmatrix} A_0 & A_1 \\ A_1 & A_0 \end{smallmatrix}\right)$ be a symmetric block matrix of order 2×2. Then $Spec(A) = Spec(A_0 + A_1) \cup Spec(A_0 - A_1)$.*

1.2 Review of literature

Earlier to 1940's, the study of graph energy was limited to the molecular graphs in chemistry. Gutman [28] was the one who initiated the study of energy of an arbitrary graph and developed a special branch in graph theory which deals with chemistry and mathematics together. He defined energy of a graph $E(G)$ as the sum of absolute values of eigenvalues of adjacency matrix of the graph. During past few years, the topic attracted many researchers from various fields due to its applications and research in this area gained tremendous momentum. It resulted in different studies on new variations in energy of a graph based on the different matrices such as Laplacian energy, incidence energy, distance energy etc [31]. In this direction, an attempt was made by Adiga et al. [3] in 2013. They introduced color energy $E_c(G)$ of a graph G based on the color matrix $A_c(G)$ of G . The authors obtained few bounds for color energy of a graph and properties of eigenvalues of color matrix. They considered the color energy of a graph with respect to minimum number of colors $\chi(G)$ and determined $E_\chi(G)$ for families of complete graphs, null graphs, star graphs, complete bipartite graphs, crown graphs and cocktail party graphs. In addition to this, another analogous concept has been introduced in [3] which is color complement of a graph and its color energy.

In past few years, researchers explored this topic and obtained color energy of different graph classes with respect to proper coloring [2, 37, 38, 56, 65], but there is a little study available on color energy of a graph with respect to various colorings and their relation. In this direction we ask, *Can we define a variation of color energy which can accommodate all the type of colorings?* Therefore, we began the present study with an attempt to determine the same.

Motivated by the studies on variety of graph energy, several new variations in color energy of graphs have been proposed into literature. The very first variation is color Laplacian energy of a graph. This concept was introduced independently by two groups of researchers, namely Bhat and D'souza [9] as well as Shigehalli and Betagiri [64] in 2015. Later, several researchers introduced various color energies like reduced color energy, color signless Laplacian energy etc [8, 10, 42, 57]. We have observed that in all these findings new dimensions of color energy have been revealed. In the present study, we make an attempt to adopt a different approach towards the color energy and widen its horizons for further research. In this view one natural question arise, *Can we establish another variation of color energy based on a different matrix?*

The inspiration to initiate the concept of color matrix arose from the concept of L-matrix introduced by Sampathkumar and Sriraj [64]. In fact, color matrix is a special

case of L-matrix. The authors defined *L-matrix* as $M = [a_{ij}]$ whose elements are

$$a_{ij} = \begin{cases} 2 & \text{if } v_i \text{ and } v_j \text{ are adjacent such that } c(v_i) = c(v_j), \\ 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent such that }]c(v_i) \neq c(v_j), \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent such that } c(v_i) = c(v_j), \\ 0 & \text{otherwise.} \end{cases}$$

Later, they denoted L-matrix of a graph G as $P_k(G)$ and introduced a variation in energy of graph by considering vertex partitions of G, namely *k-partition energy* $E_{P_k}(G)$. It is defined as the sum of the absolute values of eigenvalues of L-matrix [61]. The authors obtained bounds for $E_{P_k}(G)$ and derived properties of $P_k(G)$. Moreover, they determined $E_{P_k}(G)$ for various classes of graphs and join of regular graphs. In addition to this, the k-partition energy of generalized complements of a graph is also obtained in [61].

We have observed that, in $E_{P_k}(G)$, the properties of elements of partition are not taken into consideration. Therefore in the present study, we make an attempt to answer the question *Can we extend the concept of* $E_{P_k}(G)$ *by considering properties of elements of partition?*

After reviewing the literature, we have observed that there is a lot of scope for further investigation in the concept of color energy and its variations. Therefore, the present study focuses on the following problems:

1. What is the impact of variations in coloring protocols of a graph G on color energy of G? and how they are interconnected?
2. What are the methods by which, we accommodate all the colorings as well as properties of elements of a vertex partition of the graph under consideration, in the color matrix?

2 Results and Discussion

In this section, we showcase the results obtained during the present study. We begin with the extended results related with color energy of a graph G.

2.1 Color energy of graphs

It has been observed that $E_c(G)$ varies according to the type of coloring. Therefore, we investigate $E_c(G)$ for proper coloring as well as improper coloring. The study includes determination of color energy of a star $K_{1,n-1}$, double star $B_{q,q}$ of order $n = 2q$, a graph $\mathscr{S}$ obtained by joining pendant vertices of $K_{1,n-1}$ and $K_{r,r} - F_1$, a graph obtained by removing 1-factor from a complete bipartite graph $K_{r,r}$ as well as their color complements. Further, the relation between $E_c(G)$ and graph theoretic parameters such as chromatic number, maximum degree, domination number is obtained.

2.1.1 Bounds of color energy

Now, we attempt to establish bounds for $E_c(G)$ in terms of few well-known parameters such as the chromatic number $\chi(G)$, the independence number $\alpha(G)$, the domination number $\gamma(G)$ and maximum degree $\Delta(G)$.

Note that, m'_c denote the number of pairs of non-adjacent vertices of the given colored graph G and D is the determinant of $A_c(G)$.

We begin with some propositions. Proposition 4.1 can be obtained directly from Theorem 2.25 in [33] and Corollary 10 in [37], and Proposition 4.2 can be derived from Theorem 2.1 in [33] and Theorem 6 in [38].

Proposition 2.1. *If G is a graph with $\gamma(\overline{G}) \geq 3$, then*

$$E_{LC}(G) = E(G).$$

Proposition 2.2. *If G is a simple graph of order n, then*

$$E_c(G) \geq \frac{\gamma(G)\sqrt{2(m+m'_c)}}{n}.$$

The following results follow directly from the Theorems 2.4 and 2.5 in [3], Theorem 2.11 in [33], Theorem 6.10 in [13] and the fact that chromatic number is always less than or equal to number of vertices of corresponding graph, after careful computations.

Theorem 2.3. *Let G be a colored graph, then*

1. $E_c(G) \leq [2\chi(G)\alpha(G)(m+m'_c)]^{\frac{1}{2}}$.
2. $E_c(G) \leq \left[n\left(n^2-n+2m'_c\right)\right]^{\frac{1}{2}}$.
3. $E_c(G) \leq [2(1+\Delta(G)\gamma(G)(m+m_c\prime)]^{\frac{1}{2}}$.

Theorem 2.4. *Let G be a colored graph, then*

1. $E_c(G) \geq \{\chi(G)[\chi-1]D^{\left(\frac{2}{\chi(G)}\right)}+2(m+m'_c)\}^{\frac{1}{2}}$.
2. $E_c(G) \geq \{[\Delta(G)+\gamma(G)][\Delta(G)+\gamma(G)-1]D^{\left(\frac{2}{\Delta(G)+\gamma(G)}\right)}+2(m+m'_c)\}^{\frac{1}{2}}$.
3. $E_c(G) \geq \{2(m+m'_c)+2\gamma(G)[2\gamma(G)-1]D^{\left(\frac{1}{\gamma(G)}\right)}\}^{\frac{1}{2}}$, *provided G has no isolated vertices.*

2.1.2 Color energy corresponding to various coloring protocols

Here, we study the color energy of few classes of graphs with proper coloring such as a double star $B_{q,q}$, $B_{q,q}-e$, the color complement of $B_{q,q}$. We observe that, $E_\chi\left(B_{q,q}\right) \geq 2(q-1)$. The following theorem gives the exact value of $E_\chi(B_{q,q})$.

Theorem 2.5. *Let $B_{q,q}$ be a double star of order greater than 1. Then*

$$E_\chi\left(B_{q,q}\right) = 3q-5+\left|\frac{-(q-1)+\sqrt{q^2+10q-7)}}{2}\right|+\left|\frac{-(q-1)-\sqrt{q^2+10q-7}}{2}\right|.$$

Proof. We observe that $A_\chi\left(B_{q,q}\right)$ is of the form $\begin{pmatrix} D_0 & D_1 \\ D_1 & D_0 \end{pmatrix}$. Since the characteristic polynomial of D_0+D_1 is

$$P_\chi[(D_0+D_1),\lambda] = |\lambda I-(D_0+D_1)| = (\lambda-1)P_\chi(N,\lambda),$$

where N is a null graph of order $q-1$. Thus,

$$P_\chi[(D_0+D_1),\lambda] = (\lambda-1)^{(q-1)}[\lambda+(q-2)] \tag{3}$$

and

$$P_\chi\left[(D_0-D_1\ ),\lambda\right]=\left|\lambda I-(D_0-D_1)\right|.$$

Note that R_q and C_q denote the q^{th} row and q^{th} column respectively of this determinant. Now we perform some row and column transformations on
$P_\chi\left[(D_0-D_1\ ),\lambda\right]$ as follows. Add second row to the first row. Subtract second row from q^{th} row, for $q\geq 3$, and take the factor $(\lambda-1)$ from each q^{th} row. Add $\sum_{q\geq 3}C_q$ to second column. Thus,

$$P_\chi\left(D_0-D_1,\lambda\right)=(\lambda-1)^{(q-2)}\left[\lambda-\frac{\left[-(q-1)+\sqrt{q^2+10q-7}\right]}{2}\right]\left[\lambda-\frac{\left[-(q-1)-\sqrt{q^2+10q-7}\right]}{2}\right]. \tag{4}$$

Hence from Lemma 1.24, Equations (3) and (4),

$$Spec_\chi\left(B_{q,q}\right)=\begin{pmatrix}-(q-2) & 1 & \frac{\left[-(q-1)+\sqrt{q^2+10q-7}\right]}{2} & \frac{\left[-(q-1)-\sqrt{q^2+10q-7}\right]}{2}\\ 1 & 2q-3 & 1 & 1\end{pmatrix}$$

and

$$E_\chi\left(B_{q,q}\right)=3q-5+\left|\frac{-(q-1)+\sqrt{q^2+10q-7})}{2}\right|+\left|\frac{-(q-1)-\sqrt{q^2+10q-7}}{2}\right|.$$

□

In [37], we observed that addition of an edge in a graph changes its color energy. Now in the next theorem we examine the effect on color energy with respect to removal of an edge. We consider the graph $B_{q,q}-e$ obtained from the colored graph $B_{q,q}$ by removing the edge e joining maximum degree vertices in $B_{q,q}$. It can be verified that $E_\chi\left(B_{q,q}-e\right)\geq 2n-7$ and in next theorem, we compute the exact value of $E_\chi\left(B_{q,q}-e\right)$.

Theorem 2.6. *If $B_{q,q}$ is a double star of order $n>1$ and e is an edge joining the vertices of maximum degree. Then*

$$E_\chi\left(B_{q,q}-e\right)=3q-6+\left|\frac{-(q-1)+\sqrt{(q-1)(q+15)}}{2}\right|+\left|\frac{-(q-1)-\sqrt{(q-1)(q+15)}}{2}\right|.$$

Theorem 2.7. *Let $B_{q,q}$ be a double star and $(\overline{B_{q,q}})_c$ be its color complement. Then*

$$E_\chi(\overline{B_{q,q}})_c = 2(2n-5).$$

In the next theorem, we consider the graph $\mathscr{S}$ obtained by adding an edge between two pendant vertices of a star $K_{1,n-1}$.

Theorem 2.8. *If $\mathscr{S}$ is a colored graph and $(\overline{\mathscr{S}})_c$ is its color complement. Then*

$$E_\chi(\overline{\mathscr{S}})_c = n-4+\left|\frac{(n-4)+\sqrt{n^2+56}}{2}\right|+\left|\frac{(n-4)-\sqrt{n^2+56}}{2}\right|.$$

Now we consider a few well studied proper colorings. To begin with we examine the color energy of a graph with respect to T-coloring.

In [63], it has been proved that for each finite set T of non-negative integers containing 0, $\chi_T(G) = \chi(G)$. Therefore, the following result is true for all finite set T. We denote the color energy of a graph with respect to T-coloring by $E_T(G)$.

Theorem 2.9. *For a graph G with respect to T-coloring with a finite set T, $E_T(G) = E_\chi(G)$.*

Now, we study color energy with respect to $L(h,k)$-coloring where h and k as positive integers. We determine $E_{LC}(G)$ where G is a double star $B_{q,q}$ and color complement of few families of graphs with diameter 2 and 3. First, we present the result on color energy of a double star under $L(h,k)$-coloring.

Theorem 2.10. *Let $B_{q,q}$ be a double star with $L(h,k)$-coloring. Then*

$$E_{LC}\left(B_{q,q}\right) = n+\sqrt{n}-4.$$

Theorem 2.11. *Let $B_{q,q}$ be a double star with $L(h,k)$-coloring with $k \geq h$ and $(\overline{B_{q,q}})_c$ be its color complement. Then*

$$E_{LC}(\overline{B_{q,q}})_c = 2(n-4).$$

In the next theorem, we consider the crown graph $K_{r,r}-F_1$ of order 2n as defined in [3]. We have derived $E_{LC}(K_{r,r}-F_1)$ in [38]. Now we derive the color energy of its color complement with respect to $L(h,k)$-coloring.

Theorem 2.12. *If $K_{r,r} - F_1$ is a crown graph of order 2n with $L(h,k)$-coloring and $(\overline{K_{r,r} - F_1})_c$ is its color complement, then*

$$E_{LC}(\overline{K_{r,r} - F_1})_c = 4(n-1).$$

It has been shown in [38] that the color energy of $K_{1,n-1}$ with $L(h,k)$-coloring, $E_{LC}(K_{1,n-1})$, is same as its energy, $E(K_{1,n-1})$. Now we determine color energy of its color complement.

Theorem 2.13. *Let $K_{1,n-1}$ be a star under $L(h,k)$-coloring with $k \geq h$ and $(\overline{K_{1,n-1}})_c$ be its color complement. Then*

$$E_{LC}(\overline{K_{1,n-1}})_c = 2(n-2).$$

Remark 2.14.

1. $E_{LC}(\overline{K_{1,n-1}})_c < E_\chi(K_{1,n-1}) = E_\chi(K_n)$.
2. For the color complement $(\overline{K_n})_c$ of a complete graph K_n, $E_{LC}(\overline{K_n})_c = E_\chi(\overline{K_n})_c$ and
3. for the color complement $(\overline{N})_c$ of a null graph N, $E_{LC}(\overline{N})_c = E_\chi(N)$.

Now, lets consider the radio k-coloring of a connected graph G wherein $k = d$ and d is a diameter of G. Note that, the color energy of a graph G with respect to Radio d-coloring is denoted by $E_{Rd}(G)$. We observe the following relation between color energy of a connected graph G and its energy $E(G)$.

Theorem 2.15. *For a connected graph G with respect to Radio d-coloring, $E_{Rd}(G) = E(G)$.*

Now, we consider Hamiltonian coloring and for this coloring we got a few interesting results which describes the conditions for the relation between the color energy of a graph G with respect to Hamiltonian coloring E_H and its energy $E(G)$.

Theorem 2.16. *For a graph G which doesn't contain a Hamiltonian path, $E_H(G) = E(G)$.*

Theorem 2.17. *For a Hamiltonian connected graph G,*

$$E_H(G) = E_{\overline{\chi}}(G)$$

where $\overline{\chi}$ is the minimum number of colors used in a improper coloring.

Theorem 2.18. *For $n \geq 5$, let G be a graph obtained from the complete graph K_{n-2}, where $V(K_{n-2}) = \{u_1, u_2, \ldots, u_{n-2}\}$, by adding the two pendant edges u_1u_{n-1} and $u_{n-2}u_n$. Then $E_H(G) = E(G)$.*

Theorem 2.19. *For a complete graph K_n,*

$$E_H(K_n) = E_\chi((\overline{K_n})_c)$$

where $(\overline{K_n})_c$ is the color complement of K_n.

Theorem 2.20. *For a complete bipartite graph $K_{r,r}$, $E_H(K_{r,r}) = 4(n-1)$.*

Remark 2.21. For a complete bipartite graph $K_{r,r}$, a complete graph K_n, a star graph $K_{1,n-1}$ and a crown graph $K_{r,r} - F_1$,

$$E_H(K_{r,r}) = 2E_\chi(K_n) = 2E_\chi(K_{1,n-1}) = E_\chi(K_{r,r} - F_1).$$

Now, we consider a few improper colorings and analyse the behavior of the color energy of graphs with respect to these colorings. We begin by considering open neighborhood coloring of graphs. It is to be noted that the color energy of a graph G with respect to open neighborhood coloring is denoted by $E_{\overline{\chi}_{onc}}(G)$.

Proposition 2.22. *For a complete graph K_n, $E_{\overline{\chi}_{onc}}(K_n) = E(K_n)$.*

Proposition 2.23. *For a null graph N, $E_{\overline{\chi}_{onc}}(N) = E_\chi(N)$.*

Proposition 2.24. *For a triangular book graph $B_{3,n}$, $E_{\overline{\chi}_{onc}}(B_{3,n}) = E(B_{3,n})$.*

Proposition 2.25. *For a complete bipartite graph $K_{r,r}$, $E_{\overline{\chi}_{onc}}(K_{r,r}) = 4(n-1)$.*

Remark 2.26. For a complete bipartite graph $K_{r,r}$, a complete graph K_n, a star graph $K_{1,n-1}$ and a crown graph $K_{r,r} - F_1$,

$$E_{\overline{\chi}_{onc}}(K_{r,r}) = E_H(K_{r,r}) = 2E_\chi(K_n) = 2E_\chi(K_{1,n-1}) = E_\chi(K_{r,r} - F_1).$$

Theorem 2.27. *For a crown graph $K_{r,r} - F_1$, $E_{\overline{\chi}_{onc}}(K_{r,r} - F_1) = 2(3r-2)$.*

Theorem 2.28. *For a star graph $K_{1,n-1}$, $E_{\overline{\chi}_{onc}}(K_{1,n-1}) = 2\sqrt{n-2}$.*

Theorem 2.29. *For a graph $\mathscr{S}$ obtained by joining two pendant vertices of a star graph,*

$$E_{\overline{\chi}_{onc}}(\mathscr{S}) = 1 + |a| + |b| + |c|$$

where a, b, c are the roots of a cubic equation $\lambda^3 - \lambda^2 + (n-6)\lambda - (n-4)$.

Now, we consider another improper coloring, namely k-improper c-coloring. Following are the results obtained for color energy of some standard classes of graphs with respect to k-improper c-coloring. Note that, the color energy with respect to k-improper c-coloring is denoted by $E_{\overline{\chi}_{imp}}$.

Proposition 2.30. *If P_n is a path with respect to k-improper c-coloring, then for $k \geq 2$*

$$E_{\overline{\chi}_{imp}}(P_n) = E(P_n).$$

Proposition 2.31. *If C_n is a cycle with respect to k-improper c-coloring, then for $k \geq 2$*

$$E_{\overline{\chi}_{imp}}(C_n) = E(C_n).$$

Proposition 2.32. *If W_n is a wheel with respect to k-improper c-coloring, then for $k \geq n-1 \geq 3$*

$$E_{\overline{\chi}_{imp}}(W_n) = E(W_n).$$

Proposition 2.33. *If K_n is a complete graph with respect to k-improper c-coloring, then for $k = n-1$*

$$E_{\overline{\chi}_{imp}}(K_n) = E(K_n).$$

2.2 Generalization of color energy: $\mathscr{P}$-energy

Sampathkumar et al. [61] introduced *the k-partition energy* using the idea of a matrix called *L-matrix*, $P_k(G)$ with respect to a vertex partition P_k that uniquely represents the given graph G. *The k-partition energy, $E_{P_k}(G)$ is sum of the absolute values of k-partition eigenvalues of $P_k(G)$.* For a given graph G, the value of $E_{P_k}(G)$ varies according to different vertex partitions. It can be observed that the properties of elements in the vertex partition is not taken into consideration while determining the value of $E_{P_k}(G)$. In the present study, we consider this aspect and introduce $\mathscr{P}$-energy as a variation of k-partition energy.

Let $\mathscr{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of the vertex set $V(G)$ of a graph $G = (V,E)$. Then the *$\mathscr{P}$-matrix of G, $A_{\mathscr{P}}(G) = D(G) + P_k(G)$, where $D(G)$ is the diagonal matrix with the i^{th} diagonal entry, the cardinality of the set $V_r \in \mathscr{P}$ containing the vertex v_i.* In other words, $A_{\mathscr{P}}(G) = (a_{ij})_{n\times n}$ where

$$a_{ij} = \begin{cases} |V_r| & \text{if } i = j \text{ and } v_i \in V_r, \text{ for } r = 1,2,\ldots k \\ 2 & \text{if } v_iv_j \in E(G) \text{ with } v_i, v_j \in V_r, \\ 1 & \text{if } v_iv_j \in E(G) \text{ with } v_i \in V_r \text{ and } v_j \in V_s \text{ for } r \neq s, \\ -1 & \text{if } v_iv_j \notin E(G) \text{ with } v_i, v_j \in V_r, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of $A_{\mathscr{P}}(G)$ is denoted by $\phi_{\mathscr{P}}(G,\lambda)$ and *the eigenvalues* $\lambda_1,\lambda_2,\ldots,\lambda_n$ *of* $A_{\mathscr{P}}(G)$ *are called* $\mathscr{P}$*-eigenvalues*. If $g_1,g_2,\ldots,g_n$ are the multiplicities of $\lambda_1>\lambda_2>\ldots>\lambda_n$ respectively, then the $\mathscr{P}$-spectrum of G is

$$Spec_{\mathscr{P}}(G)=\{\lambda_1^{g_1},\lambda_2^{g_2},\ldots,\lambda_n^{g_n}\}$$

and accordingly *the* $\mathscr{P}$*-energy,* $E_{\mathscr{P}}(G)$ *is sum of the absolute values of* $\mathscr{P}$*-eigenvalues of* $A_{\mathscr{P}}(G)$.

For a given vertex partition $\mathscr{P}$ of $V(G)$, the diagonal entries of $A_{\mathscr{P}}(G)$ are positive numbers, whereas the diagonal entries of $P_k(G)$ are zeros and remaining entries of these matrices are same which belongs to the set $\{2,1,0,-1\}$. Since the absolute values of the eigenvalues of any matrix are directly proportional to the maximum value of the absolute values of entries in the given matrix, one immediate observation is that when the cardinality of every member of the given vertex partition $\mathscr{P}$ of a graph G is greater than 1, then $E_{\mathscr{P}}(G)\geq E_{P_k}(G)$.

Another interesting observation about $\mathscr{P}$-energy is that, as the order of vertex partition $\mathscr{P}$ of a given graph G increases, value of $E_{\mathscr{P}}(G)$ decreases. Hence, $\mathscr{P}$-energy of a graph G is maximum when the vertex partition $\mathscr{P}=\{V(G)\}$ and minimum when $\mathscr{P}=\{\{v_1\},\{v_2\},\ldots,\{v_n\}\}$. We call the maximum (minimum) value of $\mathscr{P}$-energy as the *robust(shear)* $\mathscr{P}$*-energy*, $E_{\mathscr{P}_r}(G)$ $(E_{\mathscr{P}_s}(G))$, similar to the concepts of robust domination energy and shear domination energy of a graph introduced by Acharya et al. [1].

Example 2.34. For a null graph H of order n, it can easily be verified that $E_{\mathscr{P}_r}(H)=n^2$ and $E_{\mathscr{P}_s}(H)=n$.

2.2.1 Properties of $A_{\mathscr{P}}(G)$ and its eigenvalues

Now, we present a few observations about $A_{\mathscr{P}}(G)$ with respect to the structure of a graph G.

Observation 2.35. Given a graph $G=(V,E)$, the following are true for its $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$.

(i) If $d(v_i)$ is the degree of $v_i\in V(G)$, then $d(v_i)$ is the number of positive off-diagonal entries of the i^{th} row corresponding to the vertex v_i in $A_{\mathscr{P}}(G)$.

(ii) The elements of $A_{\mathscr{P}}(G)$, excluding its main diagonal entries has one-one correspondence with $P_k(G)$ with respect to the same vertex partition of a given graph G.

(iii) For the matrix $A_{\mathscr{P}}(G)$,

$$tr(A_{\mathscr{P}}(G)) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{k} |V_i|^2. \tag{5}$$

(iv) If m_1 is the number of edges of a graph G whose end vertices share the same partition, m_2 is the number of edges of G whose end vertices are in different partitions and m_3 is the number of pairs of non-adjacent vertices of G within the same partition, then

$$\sum_{1\leq i<j\leq n} (a_{ij})^2 = 4m_1 + m_2 + m_3. \tag{6}$$

Theorem 2.36 characterizes $\mathscr{P}$-matrix of a graph is similar to that of the characterization of L- matrix of a labeled graph as given in [63].

Theorem 2.36. *A symmetric matrix $A = (a_{ij})_{n\times n}$ with positive diagonal entries and off-diagonal entries belonging to the set {2, 1, 0, -1} is the $\mathscr{P}$-matrix graph G of order n with the vertex partition $\mathscr{P} = \{V_1, V_2, \ldots, V_k\}$ if and only if*

(i) $a_{ij}, a_{jk} \in \{2,-1\} \implies a_{ik} \in \{2,-1\}$,

(ii) $a_{ij} \in \{2,-1\}$ *and* $a_{jk} \in \{0,1\} \implies a_{ik} \in \{0,1\}$ *and*

(iii) $v_i \in V_r \implies a_{ii} = |V_r|$.

Proposition 2.37. *If G is a graph with vertex partition $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$ and $\phi_{\mathscr{P}}(G,\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n$, then*

(i) $a_0 = 1$

(ii) $a_1 = -\sum_{i=1}^{k} |V_i|^2$

(iii) $a_2 = \sum_{1\leq i<j\leq k} |V_i||V_j| - (4m_1 + m_2 + m_3)$.

The trace of a matrix is sum of the eigenvalues of that matrix, therefore the sum of $\mathscr{P}$-eigenvalues of $A_{\mathscr{P}}(G)$ of a graph G are non-zero. In the next proposition, we obtain its value in terms of cardinality of elements in the vertex partition $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$ of G.

Proposition 2.38. *If G is a graph with the vertex partition $\mathscr{P}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the $\mathscr{P}$-eigenvalues, then*

(i) $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{k} |V_i|^2$

(ii) $\sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3.$

The next proposition follows from Cauchy-Schwartz inequality and Proposition 2.38 (ii). Note that, the symbols m_1, m_2, m_3 for graph G_1 are as given in the Observation 2.35(iv) and m_1', m_2', m_3' are the corresponding values for the graph G_2.

Proposition 2.39. *Let G_1 and G_2 be two graphs with respect to vertex partition $\mathscr{P} = \{V_1, V_2, \ldots,$*
$V_k\}$ and $\mathscr{P}' = \{V_1', V_2', \ldots, V_k'\}$ respectively. If $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $\{\lambda_1', \lambda_2', \ldots, \lambda_n'\}$ are the $\mathscr{P}$-eigenvalues of $\mathscr{P}$-matrix of G_1 and G_2 respectively, then

$$\sum_{i=1}^{n} \lambda_i \lambda_i' \leq \sqrt{\left[\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3\right]\left[\sum_{i=1}^{k} |V_i'|^3 + 8m_1' + 2m_2' + 2m_3'\right]}.$$

2.2.2 Bounds of $\mathscr{P}$-energy

Now we present some bounds for the $\mathscr{P}$-energy of a graph G in terms of its order and the cardinality of elements in its vertex partition. One obvious bound when $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$ is

$$\sum_{i=1}^{k} |V_i|^2 \leq E_{\mathscr{P}}(G) \leq n^3.$$

The lower bound follows from the inequality $\sum_{i=1}^{n} \lambda_i \leq \sum_{i=1}^{n} |\lambda_i|$ whereas the upper bound is a direct deduction from Theorem 1.21.

Theorem 2.40. *For any graph G with vertex partition $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$,*

$$E_{\mathscr{P}}(G) \leq \sqrt{n\left\{\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3\right\}}. \tag{7}$$

Proof. By Cauchy-Schwartz inequality,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{8}$$

Replace $a_i = 1$ and $b_i = |\lambda_i|$ in Equation (8),

$$\Big(\sum_{i=1}^{n}|\lambda_i|\Big)^2 \leq (\sum_{i=1}^{n}1)(\sum_{i=1}^{n}|\lambda_i|^2)$$
$$\leq n\sum_{i=1}^{n}\lambda_i^2.$$

By Proposition 2.38 (ii),

$$\Big(\sum_{i=1}^{n}|\lambda_i|\Big)^2 \leq n\Big\{\sum_{i=1}^{k}|V_i|^3 + 8m_1 + 2m_2 + 2m_3\Big\}.$$

Hence,

$$E_{\mathscr{P}}(G) \leq \sqrt{n\Big\{\sum_{i=1}^{k}|V_i|^3 + 8m_1 + 2m_2 + 2m_3\Big\}}.$$

□

Theorem 2.41. *Let G be a graph with vertex partition $\mathscr{P} = \{V_1, V_2, \ldots V_k\}$ and $|A_{\mathscr{P}}(G)|$ be the determinant of $A_{\mathscr{P}}(G)$. Then*

$$E_{\mathscr{P}}(G) \geq \sqrt{\sum_{i=1}^{k}|V_i|^3 + 8m_1 + 2m_2 + 2m_3 + n(n-1)|A_{\mathscr{P}}(G)|^{2/n}}. \tag{9}$$

Proof. By the definition of $\mathscr{P}$-energy,

$$\big[E_{\mathscr{P}}(G)\big]^2 = (\sum_{i=1}^{n}|\lambda_i|)^2 = (\sum_{i=1}^{n}|\lambda_i|)(\sum_{j=1}^{n}|\lambda_j|)$$
$$= \sum_{i=1}^{n}|\lambda_i|^2 + \sum_{i\neq j}|\lambda_i||\lambda_j|. \tag{10}$$

By using arithmetic and geometric mean inequality, Equation (10) can be written as follows

$$\big[E_{\mathscr{P}}(G)\big]^2 \geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\Big(\prod_{i\neq j}|\lambda_i||\lambda_j|\Big)^{\frac{1}{n(n-1)}}$$
$$\geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\Big(\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\Big)^{\frac{1}{n(n-1)}}.$$

Therefore,

$$\big[E_{\mathscr{P}}(G)\big]^2 \geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)\Big(\prod_{i=1}^{n}|\lambda_i|\Big)^{\frac{2}{n}}.$$

Hence from Remark 1.20(v) and Proposition 2.38 (ii),

$$E_{\mathscr{P}}(G) \geq \sqrt{\sum_{i=1}^{k} |V_i|^3 + 8m_1 + 2m_2 + 2m_3 + n(n-1)|A_{\mathscr{P}}(G)|^{2/n}}.$$

□

Remark 2.42. Let H be a null graph of order n. Then the upper and lower bounds given by Equations (7) and (9) are sharp for the vertex partition $\mathscr{P} = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ of H.

2.2.3 Exact values for $\mathscr{P}$-energy of some graph families

In this section, we examine the $\mathscr{P}$-energy of some families of graphs for the trivial partitions $\mathscr{P} = \{V(G)\}$ and $\mathscr{P} = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$ respectively. Recall that the extreme values of $\mathscr{P}$-energy is obtained with respect to these partitions and the largest value of $E_{\mathscr{P}}(G)$ denoted by $E_{\mathscr{P}_r}(G)$ is referred to as *the robust $\mathscr{P}$-energy* and the smallest denoted by $E_{\mathscr{P}_s}(G)$ is referred to as *the shear $\mathscr{P}$-energy*.

Theorem 2.43. *For the complete graph K_n,*

$$E_{\mathscr{P}_r}(K_n) = n^2 \text{ and } E_{\mathscr{P}_s}(K_n) = n.$$

Proof. Let K_n be a complete graph and $\mathscr{P} = \{V(G)\}$. The $\mathscr{P}$-matrix of K_n is

$$A_{\mathscr{P}}(K_n) = [(n-2)I + 2J]_{n\times n}$$

where J is the matrix of order $n \times n$ whose all entries are 1 and I is identity matrix of order $n \times n$. The characteristic polynomial is $\phi_{\mathscr{P}}(K_n, \lambda) = |\lambda I - A_{\mathscr{P}}(K_n)|$. Thus,

$$\begin{aligned}\phi_{\mathscr{P}}(K_n, \lambda) &= \begin{vmatrix} \lambda - n & -2 & -2 & \ldots & -2 \\ -2 & \lambda - n & -2 & \ldots & -2 \\ -2 & -2 & \lambda - n & \ldots & -2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & 2 & \ldots & \lambda - n \end{vmatrix}_{n\times n} \\ &= [\lambda - (n-2)]^{(n-1)}[\lambda - (3n-2)].\end{aligned}$$

Therefore,

$$Spec_{\mathscr{P}}(K_n) = \{(3n-2)^1, (n-2)^{(n-1)}\}.$$

Hence,

$$E_{\mathscr{P}_r}(K_n) = n^2.$$

Now, let $\mathscr{P}$ be a vertex partition of K_n such that $\mathscr{P} = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}$. The $\mathscr{P}$-matrix of K_n is

$$A_{\mathscr{P}}(K_n) = J_{n\times n}$$

and

$$\begin{aligned}\phi_{\mathscr{P}}(K_n, \lambda) &= \begin{vmatrix} \lambda-1 & -1 & -1 & \ldots & -1 \\ -1 & \lambda-1 & -1 & \ldots & -1 \\ -1 & -1 & \lambda-1 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ -1 & -1 & 1 & \ldots & \lambda-1 \end{vmatrix}_{n\times n} \\ &= \lambda^{(n-1)}(\lambda - n).\end{aligned}$$

Therefore,

$$Spec_{\mathscr{P}}(K_n) = \{n^1, 0^{(n-1)}\}.$$

Hence,

$$E_{\mathscr{P}_s}(K_n) = n.$$

□

Remark 2.44. For a complete graph K_n and a null graph H,

$$E_{\mathscr{P}_r}(K_n) = E_{\mathscr{P}_r}(H) \text{ and } E_{\mathscr{P}_s}(K_n) = E_{\mathscr{P}_s}(H).$$

Note that K_n and H are non-cospectral equi-$\mathscr{P}$-energetic graphs, since the $\mathscr{P}$-eigenvalues of $\mathscr{P}$-matrices of both the graphs differ but the values of their robust and shear $\mathscr{P}$-energies coincide.

The following result deals with the robust and shear $\mathscr{P}$-energy of a star. Its proof technique is similar to that of Theorem 2.43.

Theorem 2.45. *If $K_{1,n-1}$ is a star of order $n \geq 2$, then*

$$\begin{aligned}&E_{\mathscr{P}_r}(K_{1,n-1}) = 2n - 4 + \sqrt{n^2 + 12n - 12} \text{ and} \\ &E_{\mathscr{P}_s}(K_{1,n-1}) = (n-2) + 2\sqrt{n-1}.\end{aligned}$$

If we join the maximum degree vertex of two copies of $K_{1,r-1}$ of order $r(r \geq 2)$, then the resultant graph is called a double star $B_{r,r}$ of order $n = 2r$.

Theorem 2.46. *If $B_{r,r}$ is a double star of order n, then*

$$E_{\mathscr{P}_r}(B_{r,r}) = n^2 \quad \text{for } n \geq 2,$$
$$E_{\mathscr{P}_s}(B_{r,r}) = \begin{cases} n-1+\sqrt{2n-3} & \text{for } 2 \leq n < 8, \\ (n-4)+2\sqrt{2n-3} & \text{for } n \geq 8. \end{cases}$$

Theorem 2.47. *If $K_{r,r}$ is a complete bipartite graph of order $n = 2r \geq 2$, then*

$$E_{\mathscr{P}_r}(K_{r,r}) = n^2+2n-2 \text{ and } E_{\mathscr{P}_s}(K_{r,r}) = 2(n-1).$$

Remark 2.48. We observe that, the robust $\mathscr{P}$-energy of $K_{r,r}$, 2-partition energy of $K_{1,n-1}$ and color energy of $K_{1,n-1}$ with respect to minimum number of colors χ are same, that is $E_{\mathscr{P}_s}(K_{r,r}) = E_{P_2}(K_{1,n-1}) = E_{\chi}(K_{1,n-1})$.

Now, we proceed to determine $E_{\mathscr{P}}(G)$ for some families of graphs with respect to non-trivial vertex partitions.

Theorem 2.49. *For the star $K_{1,n-1}, n \geq 3$, with vertex partition $\mathscr{P} = \{\{v_1\}, \{v_2, v_3, \ldots, v_n\}\}$ where v_1 is the central vertex and $v_2, v_3, \ldots, v_n$ are pendant vertices of $K_{1,n-1}$,*

$$E_{\mathscr{P}}(K_{1,n-1}) = n(n-2)+2\sqrt{n-1}.$$

Now, we derive $\mathscr{P}$-energy of a double star $E_{\mathscr{P}}(B_{s,s})$ for different partitions and for that we consider, $V(B_{s,s}) = \{u_1, u_2, \ldots, u_s, v_1, v_2, \ldots, v_s\}$ such that u_1, v_1 are the maximum degree (central) vertices. Note that, the pendant vertices u_i's are attached to u_1 and the pendant vertices v_i's are attached to v_1, for $i = 2, 3, \ldots, s$.

Theorem 2.50. *If $B_{s,s}$ is a double star of order $n \geq 6$ with the vertex partition $\mathscr{P} = \{\{u_1, v_1\}, \{u_2, u_3, \ldots, u_s, v_2, v_3, \ldots, v_s\}\}$ where u_1 and v_1 are the central vertices, then*

$$E_{\mathscr{P}}(B_{s,s}) = \begin{cases} (n-4)(n-1)+\sqrt{n^2-3}+5 & \text{for } n = 6 \text{ and } 8, \\ (n-4)(n-1)+\sqrt{n^2-3}+\frac{1}{2}(5+\sqrt{2n+5}) & \text{for } n = 10, \\ (n-4)(n-1)+\sqrt{n^2-3}+\sqrt{2n+5} & \text{for } n \geq 12. \end{cases}$$

Next, we consider another partition $\mathscr{P}' = \{\{u_1, v_2, v_3, \ldots, v_s\}, \{v_1, u_2, u_3, \ldots, u_s\}\}$ of $V(B_{s,s})$ such that $\{u_1, v_2, v_3, \ldots, v_s\}$ and $\{v_1, u_2, u_3, \ldots, u_s\}$ are two independent sets where u_1, v_1 are the central vertices and $u_2, u_3, \ldots, u_s, v_2, v_3, \ldots, v_s$ are the pendent vertices of $B_{s,s}$.

Theorem 2.51. *Let $B_{s,s}$ be a double star of order $n \geq 6$ with the vertex partition $\mathscr{P}' = \{\{u_1, v_2,$
$v_3, \ldots, v_s\}, \{v_1, u_2, u_3, \ldots, u_s\}\}$ where u_1, v_1 are the central vertices of $B_{s,s}$. Then*

$$E_{\mathscr{P}'}(B_{s,s}) = \frac{1}{2}\left[n^2 - 2n - 4 + \sqrt{n^2 + 20n - 28}\right].$$

Another possibility of the vertex partition $\mathscr{P}$ having 2 elements for a $B_{s,s}$ is taking one copy of a star $K_{1,r-1}$ in each of the two elements of $\mathscr{P}$. The next result gives its corresponding $\mathscr{P}$-energy.

Theorem 2.52. *Let $B_{s,s}$ be a double star of order $n \geq 6$ with the vertex partition $\mathscr{P}'' = \{\{u_1, u_2,$
$u_3, \ldots, u_s\}, \{v_1, v_2, v_3, \ldots, v_s\}\}$ where u_1, v_1 are the central vertices of $B_{s,s}$. Then*

$$E_{\mathscr{P}''}(B_{s,s}) = \frac{1}{2}\left[n^2 - 2n - 8 + \sqrt{n^2 + 20n - 28} + \sqrt{n^2 + 28n - 60}\right].$$

Remark 2.53. From Theorems 2.50, 2.51 and 2.52, we observe that

$$E_{\mathscr{P}} \geq E_{\mathscr{P}''} \geq E_{\mathscr{P}'}.$$

Theorem 2.54. *Let $K_{r,r}$ be a complete bipartite graph of order n with a vertex partition $\mathscr{P} = \{V_1, V_2\}$ such that V_1 and V_2 are two partite sets of $K_{r,r}$. Then*

$$E_{\mathscr{P}}(K_{r,r}) = \frac{1}{2}[n^2 + 2n - 4].$$

Theorem 2.55. *Let $K_{r,r}$ be a complete bipartite graph of order n with bipartite sets $\{u_1, u_2, \ldots,$
$u_r\}$ and $\{v_1, v_2, \ldots, v_r\}$ and the vertex partition $\mathscr{P} = \{\{u_i, v_i\}$, for $1 \leq i \leq r\}$. Then*

$$E_{\mathscr{P}}(K_{r,r}) = 3n - 2.$$

Now, consider a graph obtained by removing 1-factor F_1 from a complete bipartite graph $K_{r,r}$ and denote is by $K_{r,r} - F_1$ *[5]*.

Theorem 2.56. *Let $K_{r,r} - F_1$ be a graph of order $n = 2r$, for $r \geq 3$ with a vertex partition $\mathscr{P} = \{V_1, V_2\}$ such that V_1 and V_2 are two partite sets of $K_{r,r} - F_1$. Then*

$$E_{\mathscr{P}}(K_{r,r} - F_1) = \frac{1}{2}[n^2 + 2n - 8].$$

In the next theorem, we consider the partition $\mathscr{P} = \{\{u_i, v_i\},\ \text{for } 1 \leq i \leq r\}$ and determine the corresponding $\mathscr{P}$-energy for $K_{r,r} - F_1$.

Theorem 2.57. *Let $K_{r,r} - F_1$ be a graph of order $n = 2r$, for $r \geq 3$ with a vertex partition $\mathscr{P} = \{\{u_i, v_i\},\ \text{for } 1 \leq i \leq r\}$. Then*

$$E_{\mathscr{P}}(K_{r,r} - F_1) = \begin{cases} 2n & \text{for } n = 6 \text{ and } 8, \\ 3n - 8 & \text{for } n > 8. \end{cases}$$

2.3 $\mathscr{P}$-energy of generalized Petersen graphs $G(p,k)$

In the present study, we examine the $\mathscr{P}$-energy of $G(p,k)$ with respect to vertex partitions of various types: independent, domatic, total domatic, k-ply domatic partitions, and partition containing a perfect matching in $G(p,k)$.

As evident from the definition, the value $\mathscr{P}$-energy of a graph depends on the nature of the partition $\mathscr{P}$ of the vertex set. We examine the $\mathscr{P}$-energy of the generalized Petersen graphs $G(p,k)$ by considering the trivial partitions $\mathscr{P}_r$ and $\mathscr{P}_s$, partition $\mathscr{P}(pm)$ containing a perfect matching, independent partition $\mathscr{P}(i)$, domatic partition $\mathscr{P}(d)$, total domatic partition $\mathscr{P}(td)$, and k-ply domatic partitions $\mathscr{P}(d^k)$ for $k = 2, 3$.

We begin by examining $E_{\mathscr{P}}(G)$ for the trivial partitions. It is to be noted that, the $\mathscr{P}$-energy of a graph G with vertex partition $\mathscr{P}_r = V(G)$ is the robust $\mathscr{P}$-energy $E_{\mathscr{P}_r}(G)$ and the $\mathscr{P}$-energy of G with vertex partition $\mathscr{P}_s$ containing all the singleton set of vertices is the shear $\mathscr{P}$-energy $E_{\mathscr{P}_s}(G)$ [39].

Theorem 2.58. *For the generalized Petersen graph $G(p,k)$, $E_{\mathscr{P}_r}G(p,k) = n^2$.*

In the next theorem we consider the vertex partition

$$\mathscr{P}_s = \{\{u_0\}, \{u_1\}, \{u_2\}, \ldots, \{u_{p-1}\}, \{v_0\}, \{v_1\}, \{v_2\}, \ldots, \{v_{p-1}\}\}$$

of $V(G(p,k))$ and investigate the corresponding $\mathscr{P}$-energy.

Theorem 2.59. *For the generalized Petersen graph $G(p,1)$,*

$$E_{\mathscr{P}_s}G(p,k) = \sum_{j=1}^{p} \left| 2 + 2cos\left(\frac{2\pi j}{p}\right) \right| + \sum_{l=1}^{p} \left| 2cos\left(\frac{2\pi l}{p}\right) \right|. \tag{11}$$

2.3.1 $E_{\mathscr{P}}(G(p,k))$ corresponding to independent partition $\mathscr{P}$

Now we proceed to examine $\mathscr{P}$-energy when the vertex partitions are independent sets. We consider the special case when $G(p,k)$ is bipartite such that the bi-partition $\mathscr{P}(i) = \{V_1, V_2\}$ where $V_1 = \{u_0, u_2, u_4, \ldots, u_{p-2}, v_1, v_3, v_5, \ldots, v_{p-1}\}$ and $V_2 = \{u_1, u_3, u_5, \ldots, u_{p-1}, v_0, v_2, v_4, \ldots, v_{p-2}\}$ which is in fact an independent partition. It has been proved in [52] that $G(p,k)$ is bipartite if and only if p is even and k is odd. Thus in Theorem 2.60, we determine the $\mathscr{P}$-energy of $G(p,k)$ with respect to the vertex partition $\mathscr{P}(i)$ for even p and odd k.

Theorem 2.60. *If $G(p,k)$ is a generalized Petersen graph such that $p \equiv 0(\mod 2)$ and $k = 1$ and $\mathscr{P}(i)$ is an independent vertex partition with minimum cardinality $\chi(G(p,k))$, then*

$$E_{\mathscr{P}(i)}(G(p,k)) = \sum_{l=1}^{p} \left| p + 6cos\left(\frac{2\pi l}{p}\right) + 2cos\left(\frac{6\pi l}{p}\right) \right| + \sum_{j=1}^{p} \left| p + 2 + 2cos\left(\frac{2\pi j}{p}\right) \right|. \tag{12}$$

Observation 2.61. For even values of p, it has been verified that $E_{\mathscr{P}(i)}G(p,1) = 2p^2 + 4$.

2.3.2 $E_{\mathscr{P}}(G(p,k))$ corresponding to $\mathscr{P}$ containing perfect matching

A perfect matching is a matching in a graph G such that it saturates all the vertices in G [70]. Now, we consider the perfect matching containing all spokes in $G(p,k)$ as a vertex partition $\mathscr{P}(pm)$ and compute the corresponding $\mathscr{P}$-energy.

Theorem 2.62. *If $G(p,k)$ is a generalized Petersen graph for $k = 1$ and $\mathscr{P}(pm) = \bigcup_{i=0}^{p-1} \{\{u_i, v_i\}\}$ such that its members form a perfect matching, then*

$$E_{\mathscr{P}(pm)}G(p,k) = \sum_{i=1}^{p} \left| 4 + 2cos\left(\frac{2\pi i}{p}\right) \right| + \sum_{j=1}^{p} \left| 2cos\left(\frac{2\pi j}{p}\right) \right|. \tag{13}$$

It has been verified that for $p \geq 3$ and $k = 1$, there is a beautiful relation between $E_{\mathscr{P}(pm)}G(p,k)$ and the shear $\mathscr{P}$-energy $E_{\mathscr{P}_s}G(p,k)$. However it has not yet been proved theoretically. Hence we present it as a conjecture.

Conjecture 2.63. For $p \geq 3$ and $k = 1$, $E_{\mathscr{P}(pm)}G(p,k) - E_{\mathscr{P}_s}G(p,k) = 2p$.

Now, we consider another perfect matching in $G(p,k)$ and let

$$\mathscr{P}(pm) = \left\{ \{ \bigcup_i \{u_i, u_{i+1}\}, \bigcup_i \{v_i, v_{i+1}\} \} : i \equiv 0(\mod 2), 0 \leq i \leq p-2 \right\}.$$

In the next theorem, we obtain its corresponding $\mathscr{P}$-energy. We omit its proof as it is similar to the proof of Theorem 2.62.

Theorem 2.64. *For the generalized Petersen graph $G(p,1)$ such that p is even,*

$$E_{\mathscr{P}(pm)}(G(p,1)) = \sum_{t=1}^{p} \left| 3+3cos\left(\frac{2\pi t}{p}\right) + isin\left(\frac{2\pi t}{p}\right) \right| + \sum_{l=1}^{p} \left| 1+3cos\left(\frac{2\pi l}{p}\right) + isin\left(\frac{2\pi l}{p}\right) \right|. \tag{14}$$

where $\mathscr{P}(pm) = \left\{ \{ \bigcup_i \{u_i, u_{i+1}\}, \bigcup_i \{v_i, v_{i+1}\} \} : i \equiv 0(\mod 2), 0 \leq i \leq p-2 \right\}.$

2.3.3 $E_{\mathscr{P}}(G(p,k))$ corresponding to $\mathscr{P}$ related with domination

Now, we consider vertex partitions related with the concept of domination. First, we determine the $\mathscr{P}$-energy of $G(p,k)$ with respect to the domatic partition. It can be observed that the $\mathscr{P}$-matrix of $G(p,k)$ corresponding to a domatic partition is positive semi-definite. Hence the value of $E_{\mathscr{P}(d)}(G(p,k))$ can be determined using the approach followed in Theorem 2.80.

Theorem 2.65. *For the generalized Petersen graph $G(p,k)$ having domatic partition $\mathscr{P}(d)$,*

$$E_{\mathscr{P}(d)}(G(p,k)) = p^2. \tag{15}$$

In the next theorem, we consider total domatic partition of the vertex set of $G(p,k)$ and obtain the corresponding $\mathscr{P}$-energy in the special case when $k = 1$.

Theorem 2.66. *For the generalised Petersen graph $G(p,1)$ with a total domatic partition* $\mathscr{P}(td) = \left\{ \bigcup_{i=0}^{p-1} \{u_i\}, \bigcup_{i=0}^{p-1} \{v_i\} \right\}$,

$$E_{\mathscr{P}(td)}(G(p,1)) = \begin{cases} 14 + \sum_{j=0}^{1}\sum_{s=1}^{p-1}\left| p + (-1)^{s+1} + (-1)^{j} \right. \\ \left. + 4cos\left(\frac{2s\pi}{p}\right) - 2\sum_{r=2}^{p/2-1} cos\left(\frac{2sr\pi}{p}\right)\right|, & \text{when } p \text{ is even} \\ 14 + \sum_{j=0}^{1}\sum_{s=1}^{p-1}\left| p + (-1)^{j} \right. \\ \left. + 4cos\left(\frac{2s\pi}{p}\right) - 2\sum_{r=2}^{(p-1)/2} cos\left(\frac{2sr\pi}{p}\right)\right|, & \text{when } p \text{ is odd.} \end{cases}$$

Remark 2.67. It can be verified through calculation that $E_{\mathscr{P}(td)}(G(p,1)) = 2p^2$.

The $\mathscr{P}$-matrix of $G(p,k)$ for $p > 4, k > 1$ with total domatic partition is a semi-definite matrix. Therefore, it leads to the following theorem.

Theorem 2.68. *For generalized Petersen graph $G(p,k)$ such that $p > 4, k > 1$ with the total domatic partition $\mathscr{P}(td) = \left\{ \bigcup_{i=0}^{p-1}\{u_i\}, \bigcup_{i=0}^{p-1}\{v_i\} \right\}$,*

$$E_{\mathscr{P}(td)}(G(p,k)) = 2p^2. \tag{16}$$

Now, we obtain the $\mathscr{P}$-energy of $G(p,k)$ with doubly domatic partition.

Theorem 2.69. *Let $G(p,k)$ be a generalized Petersen graph where p is even. If $\mathscr{P}(d^2)$ is a doubly domatic partition of $G(p,k)$ such that it is not a triply domatic partition and order of each member of $\mathscr{P}$ is same, then*

$$E_{\mathscr{P}(d^2)}(G(p,k)) = 2p^2. \tag{17}$$

Observation 2.70. Combining all the results of $\mathscr{P}$-energy of $G(p,1)$ corresponding to the various vertex partitions we have considered so far, we obtain the following interesting chain of inequalities.

1. If p is even and $k = 1$, then

$$\begin{aligned} &E_{\mathscr{P}_r}(G(p,k)) > E_{\mathscr{P}(i)}(G(p,k)) = E_{\mathscr{P}(d^3)}(G(p,k)) > E_{\mathscr{P}(td)}(G(p,k)) \\ &= E_{\mathscr{P}(d^2)}(G(p,k)) > E_{\mathscr{P}(d)}(G(p,k)) > E_{\mathscr{P}(pm)}(G(p,k)) > E_{\mathscr{P}_s}(G(p,k)). \end{aligned} \tag{18}$$

2. If p is odd and $k = 1$, then

$$E_{\mathscr{P}_r}(G(p,k)) > E_{\mathscr{P}(d^2)}(G(p,k)) > E_{\mathscr{P}(td)}(G(p,k)) > E_{\mathscr{P}(i)}(G(p,k)) > E_{\mathscr{P}(d)}(G(p,k)) > E_{\mathscr{P}(pm)}(G(p,k)) > E_{\mathscr{P}_s}(G(p,k)). \quad (19)$$

2.4 $\mathscr{P}$-energy of join of graphs

In the present section, we determine $\mathscr{P}$-energy for the join of two graphs using the concept of M-coronal of a graph, a method adopted in [19, 46]. We further discuss the $\mathscr{P}$-energy for the complements of the join of graphs. We have adopted the method of first finding the characteristic polynomial $\phi_{\mathscr{P}}(G)$ of G in terms of the coronals of the component graphs and then determining the value $E_{\mathscr{P}}(G)$.

2.4.1 Join of graphs

Now, we present a generalized expression for the characteristic polynomial of the graph $G_1 \triangledown G_2$ in terms of the characteristic polynomials and coronals of the component graphs. Further, we obtain $\mathscr{P}$- energy of join of graphs with respect to specific vertex partitions. Recall that $\Gamma_{A_{\mathscr{P}}}(\lambda)$ denote the coronal corresponding to the matrix $A_{\mathscr{P}}(G)$.

Theorem 2.71. *If $\mathscr{P}_1$ and $\mathscr{P}_2$ are the vertex partitions of two graphs G_1 and G_2 of order n_1 and n_2 respectively, then*

$$\phi_{\mathscr{P}}(G_1 \triangledown G_2, \lambda) = \phi_{\mathscr{P}_1}(G_1, \lambda)\phi_{\mathscr{P}_2}(G_2, \lambda)[1 - \Gamma_{A_{\mathscr{P}_1}}(\lambda)\Gamma_{A_{\mathscr{P}_2}}(\lambda)]. \quad (20)$$

Proof. For a vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1 \triangledown G_2$, the $\mathscr{P}$-matrix

$$A_{\mathscr{P}}(G_1 \triangledown G_2) = \begin{pmatrix} A_{\mathscr{P}_1}(G_1)_{n_1 \times n_1} & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & A_{\mathscr{P}_2}(G_2)_{n_2 \times n_2} \end{pmatrix}.$$

Therefore, its characteristic polynomial

$$\phi_{\mathscr{P}}(G_1 \triangledown G_2, \lambda) = \begin{vmatrix} \lambda I_{n_1} - A_{\mathscr{P}_1}(G_1)_{n_1 \times n_1} & -J_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & \lambda I_{n_2} - A_{\mathscr{P}_2}(G_2)_{n_2 \times n_2} \end{vmatrix}.$$

By Lemma 1.23,

$$\phi_{\mathscr{P}}(G_1 \triangledown G_2, \lambda) = \left|\lambda I_{n_1} - A_{\mathscr{P}_1}(G_1)\right| \left|[\lambda I_{n_2} - A_{\mathscr{P}_2}(G_2)] - B\right|. \quad (21)$$

where

$$B = J_{n_2\times n_1}[\lambda I_{n_1} - A_{\mathscr{P}_1}(G_1)]^{-1}J_{n_1\times n_2}$$

On computing the value of B, it can be written as the product of a scalar quantity s and $J_{n_2\times n_2}$, where s is the sum of all the entries from $[\lambda I_{n_1} - A_{\mathscr{P}_1}(G_1)]^{-1}$. Therefore by the definition of the $\mathscr{P}$-coronal of a graph,

$$B = \Gamma_{A_{\mathscr{P}_1}}(\lambda)J_{n_2\times n_2}. \tag{22}$$

Thus from Equations (21) and (22),

$$\phi_{\mathscr{P}}(G_1\triangledown G_2,\lambda) = \phi_{\mathscr{P}_1}(G_1,\lambda)\big|\lambda I_{n_2} - A_{\mathscr{P}_2}(G_2) - \Gamma_{A_{\mathscr{P}_1}}(\lambda)J_{n_2\times n_2}\big|.$$

By Lemma 1.19,

$$\phi_{\mathscr{P}}(G_1\triangledown G_2,\lambda) = \phi_{\mathscr{P}_1}(G_1,\lambda)\big|\lambda I_{n_2} - A_{\mathscr{P}_2}(G_2)\big|\big[1 - \Gamma_{A_{\mathscr{P}_1}}(\lambda)\Gamma_{A_{\mathscr{P}_2}}(\lambda)\big].$$

Hence, the characteristic polynomial of $A_{\mathscr{P}}(G_1\triangledown G_2)$

$$\phi_{\mathscr{P}}(G_1\triangledown G_2,\lambda) = \phi_{\mathscr{P}_1}(G_1,\lambda)\phi_{\mathscr{P}_2}(G_2,\lambda)[1 - \Gamma_{A_{\mathscr{P}_1}}(\lambda)\Gamma_{A_{\mathscr{P}_2}}(\lambda)].$$

□

Remark 2.72. For $\mathscr{P} = V(G_1\triangledown G_2)$,

$$\phi_{\mathscr{P}}(G_1\triangledown G_2,\lambda) = \phi_{\mathscr{P}_1}(A,\lambda)\phi_{\mathscr{P}_2}(B,\lambda)[1 - 4\Gamma_A(\lambda)\Gamma_B(\lambda)]. \tag{23}$$

where $A = A_{\mathscr{P}_1}(G_1) + n_2I_{n_1}$ and $B = A_{\mathscr{P}_2}(G_2) + n_1I_{n_2}$.

It has been observed in [19, 20, 46] that the adjacency matrix of a regular graph has a constant row sum, but it is interesting to observe that in case of $\mathscr{P}$-matrix, not only regular graphs but non-regular graphs can also have $\mathscr{P}$-matrices with constant row sum. Note that, row sum is the sum of all the elements in a row of the given matrix.

We now attempt to determine the exact value of $E_{\mathscr{P}}(G_1\triangledown G_2)$ in the special case when $A_{\mathscr{P}}(G_1)$ and $A_{\mathscr{P}}(G_2)$ has a constant row sums R_1 and R_2 respectively. First we prove the following lemma that gives the value of $\Gamma_{A_{\mathscr{P}}}(\lambda)$.

Lemma 2.73. *Let G be a graph of order n such that its $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$ corresponding to the partition $\mathscr{P}$ has a constant row sum R. Then*

$$\Gamma_{A_{\mathscr{P}}}(\lambda) = \frac{n}{\lambda - R}. \tag{24}$$

Using Theorem 2.71 and Lemma 2.73, we derive the characteristic polynomial of $\mathscr{P}$-matrix of join of two graphs G_1 and G_2 in the special case when $\mathscr{P}$-matrices of G_1 and G_2 have a constant row sum R_1 and R_2 respectively.

Lemma 2.74. *If G_1 and G_2 are graphs of order n_1 and n_2 having vertex partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ such that $A_{\mathscr{P}_1}(G_1)$ has a constant row sum R_1 and $A_{\mathscr{P}_2}(G_2)$ has a constant row sum R_2, then with respect to the vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1 \triangledown G_2$,*

$$\phi_{\mathscr{P}}(G_1 \triangledown G_2, \lambda) = \phi_{\mathscr{P}_1}(G_1, \lambda)\phi_{\mathscr{P}_2}(G_2, \lambda)\left[\frac{(\lambda - R_1)(\lambda - R_2) - n_1 n_2}{(\lambda - R_1)(\lambda - R_2)}\right] \tag{25}$$

where $\mathscr{P}_1$ and $\mathscr{P}_2$ are the vertex partitions of G_1 and G_2.

Theorem 2.75. *Let G_1 and G_2 be two graph of order n_1 and n_1 respectively. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be their respective vertex partitions. Then for a vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1 \triangledown G_2$*

$$\begin{aligned} E_{\mathscr{P}}(G_1 \triangledown G_2) = E_{\mathscr{P}_1}(G_1) + E_{\mathscr{P}_2}(G_2) - \sum_{i=1,2} |R_i| \\ + \frac{1}{2}\Bigg\{ \left| (R_1 + R_2) + \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - n_1 n_2)} \right| \\ + \left| (R_1 + R_2) - \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - n_1 n_2)} \right| \Bigg\} \end{aligned}$$

where R_i is a constant row sum of $A_{\mathscr{P}}(G_i)$, for $i = 1, 2$.

Remark 2.76. As observed in Section 4.2, the $\mathscr{P}$-energy of a graph G corresponding to partition $\mathscr{P} = V(G)$ is the maximum and is referred to as the robust $\mathscr{P}$-energy $E_{\mathscr{P}_r}(G)$. Therefore, from Equations (23) and (24) it follows that robust $\mathscr{P}$-energy of $G_1 \triangledown G_2$

$$\begin{aligned} E_{\mathscr{P}_r}(G_1 \triangledown G_2) = \sum_{i=1}^{n_1} |\lambda_i + n_2| + \sum_{i=1}^{n_2} |\lambda_i' + n_1| - \sum_{i=1,2} |R_i'| \\ + \frac{1}{2}\Bigg\{ \left| (R_1' + R_2') + \sqrt{(R_1' + R_2')^2 - 4(R_1' R_2' - 4n_1 n_2)} \right| \\ + \left| (R_1' + R_2') - \sqrt{(R_1' + R_2')^2 - 4(R_1' R_2' - 4n_1 n_2)} \right| \Bigg\} \end{aligned}$$

where λ_i, λ_i' are the eigenvalues of $A_{\mathscr{P}_1}(G_1)$, $A_{\mathscr{P}_2}(G_2)$ respectively.

In the next theorem, we derive an expression for the characteristic polynomial for the join of k-copies of a graph F in terms of characteristic polynomial of F. Further, using this expression we obtain $E_{\mathscr{P}}(G)$ where G is the join of k-copies of F when $A_{\mathscr{P}}(F)$ has constant row sum.

Theorem 2.77. *Let F be a graph of order t and $\mathscr{P}_1$ be its vertex partition. If G is the join of k-copies of F, then for the vertex partition $\mathscr{P}$ of G having k elements, each of which is a $\mathscr{P}_1$,*

$$\phi_{\mathscr{P}}(G) = \left[\phi_{\mathscr{P}_1}(F,\lambda)\right]^k \left[1-(k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]\left[1+\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]^{(k-1)}. \tag{26}$$

Proof. By the choice of $\mathscr{P}$, we have the $\mathscr{P}$-matrix of G,

$$A_{\mathscr{P}}(G) = \begin{pmatrix} A_{\mathscr{P}_1}(F) & J & J & \dots & J \\ J & A_{\mathscr{P}_1}(F) & J & \dots & J \\ J & J & A_{\mathscr{P}_1}(F) & \dots & J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \dots & A_{\mathscr{P}_1}(F) \end{pmatrix}_{k\times k}.$$

Therefore, the characteristic polynomial of $A_{\mathscr{P}}(G)$

$$\phi_{\mathscr{P}}(G) = \left|\lambda I - A_{\mathscr{P}}(G)\right|.$$

Apply the following row and column operations on $\phi_{\mathscr{P}}(G)$,

(i) Add $\sum_{i=2}^{k} C_i$ to first column,

(ii) Subtract first row from i^{th} row, for $i = 2,3,\dots,k$.

Therefore on simplifying, it becomes

$$\phi_{\mathscr{P}}(G,\lambda) = \left|\lambda I - A_{\mathscr{P}_1}(F) - (k-1)J\right| \left|\lambda I - A_{\mathscr{P}_1}(F) + J\right|^{(k-1)}.$$

By Lemma 1.19,

$$\begin{aligned} \phi_{\mathscr{P}}(G,\lambda) &= \left[1-(k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]\left|\lambda I - A_{\mathscr{P}_1}(F)\right| \\ &\quad \left\{\left[1+\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]\left|\lambda I - A_{\mathscr{P}_1}(F)\right|\right\}^{(k-1)} \\ &= \left[\phi_{\mathscr{P}_1}(F,\lambda)\right]^k \left[1-(k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]\left[1+\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]^{(k-1)}. \end{aligned}$$

□

Remark 2.78. For $\mathscr{P} = V(G)$,

$$\phi_{\mathscr{P}}(G) = \left[\phi_{\mathscr{P}_1}(C,\lambda)\right]^k \left[1-2(k-1)\Gamma_C(\lambda)\right]\left[1+2\Gamma_C(\lambda)\right]^{(k-1)}$$

where $C = A_{\mathscr{P}_1}(F) + (n-t)I_t$.

Now, we consider the join of k-copies of a graph F wherein the $\mathscr{P}$-matrix of F has a constant row sum and obtain the corresponding characteristic polynomial in the the next result.

Lemma 2.79. *Let G be the join of k-copies of a graph F of order t. Let $\mathscr{P}$ and $\mathscr{P}_1$ be the vertex partitions of G and F respectively. If $A_{\mathscr{P}_1}(F)$ has constant row sum R, then for $\mathscr{P}$ such that it is the union of k-copies of $\mathscr{P}_1$,*

$$\phi_{\mathscr{P}}(G,\lambda) = \left[\frac{\phi_{\mathscr{P}_1}(F)}{\lambda - R}\right]^k \big[\lambda - R - (k-1)t\big]\big[\lambda - R + t\big]^{(k-1)}.$$

By using Lemma 2.79, we obtain the corresponding $\mathscr{P}$-energy.

Theorem 2.80. *Let F be a graph of order t and G be the join of k-copies of F. Let $\mathscr{P}_1$ and $\mathscr{P}$ be the vertex partitions of G and F respectively. If $A_{\mathscr{P}_1}(F)$ has a constant row sum R, then for a vertex partition $\mathscr{P}$ such that it is the union of k-copies of $\mathscr{P}_1$,*

$$E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(F) - kR + |R + (k-1)t| + (k-1)|R - t|. \tag{27}$$

Remark 2.81. The robust $\mathscr{P}$-energy of join G of k-copies of a graph F

$$E_{\mathscr{P}_r}(G) = k\sum_{i=1}^{t} |\lambda_i + n - t| - kR' + |R' + 2(k-1)t| + (k-1)|R' - 2t|$$

where λ_i is an eigenvalue of $A_{\mathscr{P}_1}(F)$ for $i = 1,2,\ldots,t$ and R' is the constant row sum of $A_{\mathscr{P}_1}(F) + (n-t)I_t$.

2.4.2 Generalized complements of join of graphs

In this section, we determine the $\mathscr{P}$-energy of complement and generalized complements of join of graphs. It is to be noted that, the disjoint union of graphs is the complement of the join of those graphs [58].

We begin with the $\mathscr{P}$-energy of complement of join of k-copies of a graph.

Theorem 2.82. *Let G be a graph of order n obtained by the join of k-copies of a graph F of order t and $\mathscr{P}$ be its vertex partition such that it is the union of k-copies of vertex partition $\mathscr{P}_1$ of F. If $\overline{G}$ is the complement of G, then for the vertex partition $\mathscr{P}$ of $\overline{G}$,*

$$E_{\mathscr{P}}(\overline{G}) = \sum_{i=1}^{k} E_{\mathscr{P}_i}(\overline{F}).$$

In the next theorem, we determine the characteristic polynomial of complement of the join of k-copies of F. It's proof is similar to that of Theorem 2.77.

Theorem 2.83. *Let F be a graph of order t and $\mathscr{P}_1$ be its vertex partition such that $\mathscr{P}_1 = V(\overline{F})$. Let G be a graph of order n obtained by the join of k-copies of F and $\mathscr{P}$ be its vertex partition. Then for $\mathscr{P} = V(\overline{G})$,*

$$\phi_{\mathscr{P}}(\overline{G}) = \left[\phi_{\mathscr{P}_1}(D,\lambda)\right]^k \left[1+(k-1)\Gamma_D(\lambda)\right]\left[1-\Gamma_D(\lambda)\right]^{(k-1)}.$$

where $D = A_{\mathscr{P}_1}(\overline{F}) + (n-t)I_t$.

Now we consider a graph F whose $\mathscr{P}$-matrix has a constant row sum and find the corresponding $\mathscr{P}$-energy using Theorem 2.83.

Theorem 2.84. *Let G be a graph of order n obtained by the join of k-copies of a graph F of order t such that $\mathscr{P}$ and $\mathscr{P}_1$ are their vertex partitions respectively. Let $\overline{G}$ be the complement of G If $A_{\mathscr{P}_1}(\overline{F})$ has constant row sum R, then for $\mathscr{P} = V(\overline{G})$,*

$$E_{\mathscr{P}}(\overline{G}) = k\sum_{i=1}^{t}|\lambda_i + n - t| - kR' + |R' - (k-1)t| + (k-1)|R' + t|. \tag{28}$$

Now, we obtain the $\mathscr{P}$-energy of complement of join of k graphs in the next result. We state it without proof as it's proof is similar to the proof of Theorem 2.82.

Theorem 2.85. *Let G be the join of k graphs $G_1, G_2, \ldots, G_k$ such that $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_k$ are the vertex partitions of $G_1, G_2, \ldots, G_k$ respectively and $\mathscr{P} = \bigcup_{i=1}^{k} \mathscr{P}_i$ be the vertex partition of G. If $\overline{G}$ is the complement of G, then for the vertex partition $\mathscr{P} = \bigcup_{i=1}^{k} \mathscr{P}_i$ of $V(\overline{G})$,*

$$E_{\mathscr{P}}(\overline{G}) = E_{\mathscr{P}_1}(G_1) + E_{\mathscr{P}_2}(G_2) + \ldots + E_{\mathscr{P}_k}(G_k).$$

In the next theorem, we determine the $\mathscr{P}$-energy of k-complement of join of k graphs.

Theorem 2.86. *For a graph $G = G_1 \bigtriangledown G_2 \bigtriangledown \ldots \bigtriangledown G_k$ of order n such that $\mathscr{P} = \sum_{i=1}^{k} \mathscr{P}_i$,*

$$E_{\mathscr{P}}(\overline{(G)_k}) = \sum_{i=1}^{k} E_{\mathscr{P}_i}(\overline{(G_i)_k}). \tag{29}$$

where $\overline{(G)_k}$ is the k-complement of the graph G and $\mathscr{P}_i$ is the vertex partition of G_i, for $i = 1, 2, \ldots, k$.

Remark 2.87. Let G be the join of k graphs having the vertex partition $\mathscr{P} = V(G)$ and $\overline{(G)_1}$ be the 1-complement of G. Then the robust $\mathscr{P}$-energy of $\overline{(G)_1}$

$$E_{\mathscr{P}_r}(\overline{(G)_1}) = E_{\mathscr{P}_r}(G).$$

Now, we determine the characteristic polynomial of $k(i)$-complement of join of k-copies of a graph.

Theorem 2.88. *Let F be a graph of order t and let $\mathscr{P}_1$ be its vertex partition. If G is a graph of order n obtained by the join of k-copies of F such that $\mathscr{P}$ is its vertex partition, then for $\mathscr{P}$ of $\overline{(G)_{k(i)}}$ such that it is the union of k-copies of $\mathscr{P}_1$,*

$$\phi_{\mathscr{P}}(\overline{(G)_{k(i)}}) = \left[\phi_{\mathscr{P}_1}(\overline{(F)_{k(i)}}, \lambda)\right]^k \left[1-(k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right] \left[1+\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]^{(k-1)}. \tag{30}$$

Now in Theorem 2.89 by considering the case when $A_{\mathscr{P}_1}(F)$ has a constant row sum R, we obtain the the corresponding $\mathscr{P}$-energy by the similar method as shown in the proof of Theorem 2.84.

Theorem 2.89. *Let G be the join of k-copies of a graph F of order t having $\mathscr{P}$ and $\mathscr{P}_1$ as their vertex partitions respectively. If $A_{\mathscr{P}_1}(F)$ has constant row sum R, then for a vertex partition $\mathscr{P}$ of $\overline{(G)_{k(i)}}$ such that it is the union of k-copies of vertex partitions $\mathscr{P}_1$ of $\overline{(F)_{k(i)}}$,*

$$E_{\mathscr{P}}(\overline{(G)_{k(i)}}) = kE_{\mathscr{P}_1}(\overline{(F)_{k(i)}}) - kR + |R+(k-1)t| + (k-1)|R-t|.$$

The robust $\mathscr{P}$-energy of the $k(i)$-complement of join of k graphs is given by the following result.

Proposition 2.90. *Let G_i be graphs of order n_i for $i = 1,2,\ldots,k$. If $G = G_1 \bigtriangledown G_2 \bigtriangledown \ldots \bigtriangledown G_k$, then for the vertex partition $\mathscr{P} = V(\overline{(G)_{1(i)}})$*

$$E_{\mathscr{P}_r}(\overline{(G)_{1(i)}}) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left|\lambda_{i_j} + (n-n_i)\right|$$

where $\overline{(G)_{1(i)}}$ is the $1(i)$-complement of the graph G of order n and λ_{i_j} is an eigenvalue of $A_{\mathscr{P}_i}(\overline{(G_i)_{1(i)}})$ for $j = 1,2,\ldots,n_i$ where $i = 1,2,\ldots,k$.

2.4.3 Some special cases of join of graphs

In this section, we explore the $\mathscr{P}$-energy of join of regular graphs. We begin with a lemma that gives a property of $A_{\mathscr{P}}(G)$ when G is a regular graph.

Lemma 2.91. *If G is a regular graph with vertex partition $\mathscr{P} = \{V_1, V_2, \ldots, V_k\}$ such that $|V_i| = l$, for $i = 1, 2, \ldots, k$, then $A_{\mathscr{P}}(G)$ has a constant row sum $R = l + \frac{1}{n}(4m_1 + 2m_2 - 2m_3)$ where m_1 is the number of edges having end vertices are in same vertex partition, m_2, the number of edges with end vertices are in different partition and m_3 is the number of non-adjacent pairs of vertices.*

Note that, the quantities m_1, m_2, m_3, and m_1', m_2', m_3' used in Theorem 2.92 are taken in the same context as that of m_1, m_2 and m_3 mentioned in Lemma 2.91.

Theorem 2.92. *Let G_1 and G_2 be two regular graphs of order n_1 and n_2 with vertex partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ respectively. If $\mathscr{P}_1 = \{U_1, U_2, \ldots, U_{k_1}\}$ and $\mathscr{P}_2 = \{V_1, V_2, \ldots, V_{k_2}\}$ such that $|U_i| = l_1$ and $|V_i| = l_2$, then for the vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1 \triangledown G_2$*

$$E_{\mathscr{P}}(G_1 \triangledown G_2) = \sum_{i=1,2} E_{\mathscr{P}_i}(G_i) - \sum_{i=1,2} \left| l_i + \frac{M_i}{n_i} \right| + \frac{1}{2}\left(|a_1| + |b_1|\right),$$

where $M_1 = 4m_1 + 2m_2 - 2m_3$ and $M_2 = 4m_1' + 2m_2' - 2m_3'$, $a_1 = \left[l_1 + l_2 + \frac{M_1}{n_1} + \frac{M_2}{n_2} \right] + \left\{ (l_1 - l_2)^2 + \left(\frac{M_1}{n_1} - \frac{M_2}{n_2} \right) + 2l_1 \left(\frac{M_1}{n_1} - \frac{M_2}{n_2} \right) + 2l_2 \left(\frac{M_2}{n_2} - \frac{M_1}{n_1} \right) + 4n_1 n_2 \right\}^{\frac{1}{2}}$ and $b_1 = \left[l_1 + l_2 + \frac{M_1}{n_1} + \frac{M_2}{n_2} \right] - \left\{ (l_1 - l_2)^2 + \left(\frac{M_1}{n_1} - \frac{M_2}{n_2} \right) + 2l_1 \left(\frac{M_1}{n_1} - \frac{M_2}{n_2} \right) + 2l_2 \left(\frac{M_2}{n_2} - \frac{M_1}{n_1} \right) + 4n_1 n_2 \right\}^{\frac{1}{2}}$.

Corollary 2.93. *Let G_1 and G_2 be two regular graphs of order n_1 and n_2,with degrees r_1 and r_2 respectively. If $\mathscr{P}_1$ and $\mathscr{P}_2$ are the vertex partitions of G_1 and G_2, then corresponding to the vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1 \triangledown G_2$, then,*

(i)

$$E_{\mathscr{P}}(G_1 \triangledown G_2) = \sum_{i=1,2} E_{\mathscr{P}_i}(G_i) - \sum_{i=1,2} (3r_i + 1) + \frac{1}{2}\big\{|a| + |b|\big\},$$

where $a = [3(r_1 + r_2) + 2] + \sqrt{3(r_1 - r_2)^2 + 4n_1 n_2}$ and
$b = [3(r_1 + r_2) + 2] - \sqrt{3(r_1 - r_2)^2 + 4n_1 n_2}$,
provided $\mathscr{P}_1 = \{V(G_1)\}$ and $\mathscr{P}_2 = \{V(G_2)\}$.

(ii)

$$E_{\mathscr{P}}(G_1 \triangledown G_2) = \sum_{i=1,2} E_{\mathscr{P}_i}(G_i) - \sum_{i=1,2} (2r_i - n_i + 2) + \frac{1}{2}\big\{|a_2| + |b_2|\big\},$$

where $a_2 = [2(r_1+r_2)-(n_1+n_2)+4]+\{(2r_1+2r_2)^2+(n_1+n_2)^2+4r_1(n_2-n_1)+4r_2(n_1-n_2)\}^{\frac{1}{2}}$ $b_2 = [2(r_1+r_2)-(n_1+n_2)+4]-\{(2r_1+2r_2)^2+(n_1+n_2)^2+4r_1(n_2-n_1)+4r_2(n_1-n_2)\}^{\frac{1}{2}}$ *when* $\mathscr{P}_1 = \{\{u_1\},\{u_2\}$ $,\ldots,\{u_{n_1}\}\}$ *and* $\mathscr{P}_2 = \{\{v_1\},\{v_2\},\ldots,\{v_{n_2}\}\}$.

By Theorem 2.77 and Lemma 2.91, we obtain the following result.

Theorem 2.94. *Let H be a regular graph of order n_1 and $\mathscr{P} = \{V_1, V_2, \ldots, V_t\}$ be its vertex partition such that $|V_i| = l_1$ for $i = 1,2,\ldots,t$. Let G be the join of k-copies of H and $\mathscr{P}$ be its vertex partition such that it is the union of k-copies of $\mathscr{P}$. Then for $\mathscr{P}$*

$$E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(H) + \frac{1}{n_1}|(k-1)n_1^2 + l_1 + M_1| + \frac{(k-1)}{n_1}|M_1 + l_1n_1 - n_1^2| - \frac{k}{n_1}|l_1n_1 + M_1|.$$

Corollary 2.95. *Let H be an r_1-regular graph of order n_1 and $\mathscr{P}$ be its vertex partition. Let G be the join of k-copies of H. If $\mathscr{P}$ is the vertex partition of G such that it is the union of k-copies of $\mathscr{P}$, then*

(i) $E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(H) + |3r_1 + 1 - (k-1)n_1| + (k-1)|3r_1 + 1 - n_1| - k|3r_1 + 1|$ *when* $\mathscr{P}_1 = V(H)$ *and*

(ii) $E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(H) + |2(r_1+1) - kn_1| + 2(k-1)|r + 1 - n_1| - k|2r_1 - n_1 + 2|$ *if* $\mathscr{P}_1 = \{\{u_1\},\{u_2\},\ldots,\{u_{n_1}\}\}$.

Let G be the join of k regular graphs of order $n_1, n_2, \ldots, n_k$ and degree $r_1, r_2, \ldots, r_k$ respectively. Then the join G is a regular graph if $n_i - r_i = n_{i+1} - r_{i+1}$ [20]. In the next theorem, we use this condition and obtain the $\mathscr{P}$-energy of a regular graph which is the join of k graphs, each of which is regular.

Theorem 2.96. *Let $G_1, G_2, \ldots, G_k$ be the regular graphs of order $n_1, n_2, \ldots, n_k$ and degree $r_1, r_2, \ldots, r_k$ respectively. Let $\mathscr{P}_1, \mathscr{P}_2, \ldots, \mathscr{P}_k$ be their vertex partitions. If $G = G_1 \triangledown G_2 \triangledown \ldots \triangledown G_k$ is an r-regular graph of order n such that $r = n - s$, then for a vertex partition $\mathscr{P} = \bigcup_{i=1}^{k} \mathscr{P}_i$ of G*

$$E_{\mathscr{P}}(G) = \sum_{i=1}^{k} E_{\mathscr{P}_i}(G_i) - \sum_{i=1}^{k} R_i + \left(\sum_{i=1}^{k-1} n_i + R_k\right) + \sum_{i=1}^{k-1} |R_i - n_i| \tag{31}$$

where R_i is a constant row sum of $A_{\mathscr{P}_i}(G_i)$ and $s = n_i - r_i = n_{i+1} - r_{i+1}$, for $i = 1,2,\ldots,k$.

Corollary 2.97. *Let G_i be a regular graph of order n_i, degree r_i such that $\mathscr{P}_i = V(G_i)$, for $i = 1,2,\ldots,k$. If $G = \bigcup_{i=1}^{k} G_i$ is a regular graph of degree r such that its vertex partition $\mathscr{P} = \bigcup_{i=1}^{k} \mathscr{P}_i$, then*

$$E_{\mathscr{P}}(G) = \sum_{i=1}^{k} E_{\mathscr{P}_i}(G_i) - \sum_{i=1}^{k}(3_{r_i}+1) + \left(\sum_{i=1}^{k-1} n_i + 3r_k + 1\right) + \sum_{i=1}^{k-1} |3r_i + 1 - n_i|. \quad (32)$$

In this section, we determine the $\mathscr{P}$-energy of join of complete bipartite graphs using Theorem 2.71 and 2.77. First we obtain the $\mathscr{P}$-coronal of a complete bipartite graph.

Lemma 2.98. *Let $K_{r,s}$ be a complete bipartite graph of order $n = r+s$ such that $r \neq s$. Then for the vertex partition $\mathscr{P} = \{V_1, V_2\}$ where V_1 and V_2 are the two partite sets of $K_{r,s}$ and $\mathscr{P} = \{\{v_1\},\{v_2\},\ldots,\{v_n\}\}$, the $\mathscr{P}$-coronal of $K_{r,s}$ is*

$$\Gamma_{A_{\mathscr{P}}}(\lambda) = \frac{n\lambda + 2rs - n}{\lambda^2 - 2\lambda - (rs-1)}. \quad (33)$$

Theorem 2.99. *Let K_{r_1,s_1} and K'_{r_2,s_2} be two complete bipartite graphs of order n_1 and n_2. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be their vertex partitions respectively. Then for a vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $K_{r_1,s_1} \triangledown K'_{r_2,s_2}$,*

$$E_{\mathscr{P}}(K_{r_1,s_1} \triangledown K'_{r_2,s_2}) = E_{\mathscr{P}_1}(K_{r_1,s_1}) + E_{\mathscr{P}_2}(K'_{r_2,s_2}) - |(1+\sqrt{r_1 s_1})| - |(1-\sqrt{r_1 s_1})| - |(1+\sqrt{r_2 s_2})| - |(1-\sqrt{r_2 s_2})| + |c_1| + |c_2| + |c_3| + |c_4|$$

where c_1, c_2, c_3 and c_4 are the roots of the quartic polynomial
$\lambda^4 - 4\lambda^3 - [r_1 s_1 + r_2 s_2 + n_1 n_2 - 6]\lambda^2 + 2[r_1 s_1(1-n_2) + r_2 s_2(1-n_1) + n_1 n_2 - 2]\lambda + [r_1 s_1(2n_2 - 1) + r_2 s_2(2n_1 - 1) - 3r_1 r_2 s_1 s_2 - n_1 n_2 + 1]$.

Theorem 2.100. *Let G_1 be a graph of order n_1 such that $A_{\mathscr{P}_1}(G_1)$ has a constant row sum R_1 and K_{r_1,s_1} be a complete bipartite graph of order n_2. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be their vertex partitions respectively. Then for a vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1 \triangledown K_{r_1,s_1}$,*

$$E_{\mathscr{P}}(G_1 \triangledown K_{r_1,s_1}) = E_{\mathscr{P}_1}(G_1) + E_{\mathscr{P}_2}(K_{r_1,s_1}) - |R_1| - |(1+\sqrt{r_1 s_1})| + |d_1| + |d_2| + |d_3|$$

where d_1, d_2 and d_3 are the roots of the cubic polynomial
$\lambda^3 - (R_1+2)\lambda^2 - [2R_1 - r_1 s_1 - n_1 n_2 + 1]\lambda + [R_1(r_1 s_1 - 1) - 2r_1 s_1 s_2 n_1 + n_1 n_2]$.

Now, we derive the expression for $\mathscr{P}$-energy of join of k-copies of $K_{r,s}$ in the next theorem.

Theorem 2.101. *Let $K_{r,s}$ be a complete bipartite graph of order t, for $r \neq s$ and $\mathscr{P}_1$ be its vertex partition. Let G be the join of k-copies of $K_{r,s}$ and $\mathscr{P}$ be its vertex partition such that it is the union of k-copies of $\mathscr{P}_1$. Then*

$$\begin{aligned} E_{\mathscr{P}}(G) = {} & kE_{\mathscr{P}_1}(K_{r,s}) - k(1+\sqrt{rs}) - k(1-\sqrt{rs}) \\ & + \frac{1}{2}\Bigg\{ \left| t(k-1) + 2 + \sqrt{t^2(k-1)^2 + 4[2(k-1)+1]rs} \right| \\ & + \left| t(k-1) + 2 - \sqrt{t^2(k-1)^2 + 4[2(k-1)+1]rs} \right| \\ & + \left| t - 2 + \sqrt{t^2 + 4rs} \right| + \left| t - 2 - \sqrt{t^2 + 4rs} \right| \Bigg\}. \end{aligned}$$

2.5 A programming approach

In today's world of technology, it is necessary to view the technical aspect of any topic under consideration. In this section, we review the findings in the present study from a programming perspective. We adopt a programming approach in the present study and choose python programming language to do so. Python is a powerful programming language. It is wildly used in graph theory and its applications.

2.5.1 A python program to determine $E_{\mathscr{P}}(G(p,k))$

Here, we present a python program to determine the values of $E_{\mathscr{P}}G(p,k)$ for any value of p and k, and four particular cases of partition $\mathscr{P}$: the trivial partitions $\mathscr{P}_r$ and $\mathscr{P}_s$, independent partition $\mathscr{P}(i)$ and partition $\mathscr{P}(pm)$ with perfect matching containing all spokes in $G(p,k)$.

```
import networkx as nx
import matplotlib.pyplot as plt
import numpy as np
import numpy.linalg
def Graph(n):
G=nx.circular_ladder_graph(n)
return G
def Robust_Partition(n):
G0=Graph(n)
adj_mat = nx.adjacency_matrix(G0)
x=nx.to_numpy_matrix(G0)
mat = []
for i in range (0,2*n):
mat.append([])
for i in range (0,2*n):
for j in range (0,2*n):
mat[i].append(j)
mat[i][j]=0
for i in range (0,2*n):
for j in range (0,2*n):
if (i==j):
mat[i][j] = 2*n
else:
if (x.item((i,j)) ==1):
mat[i][j] = 2
else:
mat[i][j] = -1
P_matrix=np.array(mat)
return mat
def Shear_partition(n):
G1=Graph(n)
adj_mat = nx.adjacency_matrix(G1)
x=nx.to_numpy_matrix(G1)
smat = []
for i in range (0,2*n):
smat.append([])
for i in range (0,2*n):
for j in range (0,2*n):
smat[i].append(j)
smat[i][j]=0
for i in range (0,2*n):
for j in range (0,2*n):
if (i==j):
smat[i][j] = 1
else:
if (x.item((i,j)) ==1):
smat[i][j] = 1
```

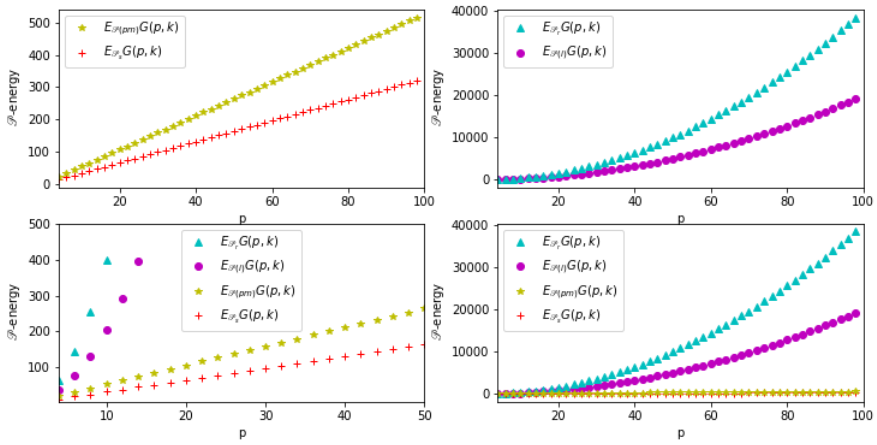
```
else:
smat[i][j] = 0
P_s_matrix=np.array(smat)
return smat
def independent_partition(n):
G2=Graph(n)
colors_m=[]
for i in range (0,n) :
if (i %2)==0:
colors_m.append('blue')
else :
colors_m.append('red')
for i in range (n,2*n) :
if (i %2)==0:
colors_m.append('red')
else :
colors_m.append('blue')
adj_mat = nx.adjacency_matrix(G2)
y=nx.to_numpy_matrix(G2)
imat = []
for i in range (0,2*n):
imat.append([])
for i in range (0,2*n):
for j in range (0,2*n):
imat[i].append(j)
imat[i][j]=0
for i in range (0,2*n):
for j in range (0,2*n):
if (i==j):
imat[i][j] = n
elif (y.item((i,j)) ==1):
imat[i][j] = 1
elif colors_m[i]==colors_m[j]:
imat[i][j] = -1
else:
imat[i][j] = 0
P_I_matrix=np.array(imat)
matrix=np.matrix(P_I_matrix)
return matrix
def spokes_partition(n):
G3=Graph(n)
colors_m=[]
for i in range (0,n) :
colors_m.append(i+1)
for i in range (n,2*n) :
colors_m.append(i-(n-1))
adj_mat = nx.adjacency_matrix(G3)
```

```
z=nx.to_numpy_matrix(G3)
pmat = []
for i in range (0,2*n):
pmat.append([])
for i in range (0,2*n):
for j in range (0,2*n):
pmat[i].append(j)
pmat[i][j]=0
for i in range (0,2*n):
for j in range (0,2*n):
if (i==j):
pmat[i][j] = 2
elif (z.item((i,j)) ==1) and colors_m[i]==colors_m[j]:
pmat[i][j] = 2
elif (z.item((i,j)) ==1) and colors_m[i]!=colors_m[j]:
pmat[i][j] = 1
else:
pmat[i][j] = 0
P_p_matrix=np.array(pmat)
matrix1=np.matrix(P_p_matrix)
return matrix1
def robust_energy(n):
a=Robust_Partition(n)
spec=np.linalg.eigvals(a)
spec_abs=np.abs(spec)
E=np.sum(spec_abs)
return E
def shear_energy(n):
b=Shear_partition(n)
spec_s=np.linalg.eigvals(b)
spec_abs_s=np.abs(spec_s)
E_s=np.sum(spec_abs_s)
return E_s
def independent_energy(n):
c=independent_partition(n)
spec_i=np.linalg.eigvals(c)
spec_abs_i=np.abs(spec_i)
E_i=np.sum(spec_abs_i)
return E_i
def spokes_energy(n):
d=spokes_partition(n)
spec_p=np.linalg.eigvals(d)
spec_abs_p=np.abs(spec_p)
E_p=np.sum(spec_abs_p)
return E_p
def plot():
rb_e = []
```

```
sh_e = []
in_e = []
sp_e = []
x_axis = []
x_range=int(48)
xmin= int(4)
xmax=int(xmin+2*x_range)
for i in range (0,x_range):
x_axis.append([])
rb_e.append([])
sh_e.append([])
in_e.append([])
sp_e.append([])
for i in range (xmin,xmax,2):
index=int((i-4)/2)
x_axis[index].append(i)
e1 = robust_energy(i)
rb_e[index].append(e1)
e2 = shear_energy(i)
sh_e[index].append(e2)
e3 = independent_energy(i)
in_e[index].append(e3)
e4 = spokes_energy(i)
sp_e[index].append(e4)
t = np.arange(xmin, xmax, 2)
plt.figure(figsize=(12,6))
plt.subplot(221)
sh_e_plot1, = plt.plot(t,sh_e,'r+')
sp_e_plot1, = plt.plot(t,sp_e,'y*')
plt.legend(handles = [sp_e_plot1,sh_e_plot1],labels =['$E_{\mathscr{P
    }(pm)}G(p,k)$','$E_{\mathscr{P}_s}G(p,k)$'])
plt.xlim(xmin,xmax)
plt.xlabel('p')
plt.ylabel(' $\mathscr{P}$-energy ')
plt.subplot(222)
in_e_plot, = plt.plot(t,in_e,'mo')
rb_e_plot, = plt.plot(t,rb_e,'c^')
plt.legend(handles = [rb_e_plot,in_e_plot],labels =['$E_{\mathscr{P}
    _r}G(p,k)$','$E_{\mathscr{P}(i)}G(p,k)$'])
plt.xlim(xmin,xmax)
plt.xlabel('p')
plt.ylabel(' $\mathscr{P}$-energy ')
plt.subplot(223)
in_e_plot, = plt.plot(t,in_e,'mo')
rb_e_plot, = plt.plot(t,rb_e,'c^')
sh_e_plot, = plt.plot(t,sh_e,'r+')
sp_e_plot, = plt.plot(t,sp_e,'y*')
```

```
plt.legend(handles = [rb_e_plot,in_e_plot,sp_e_plot,sh_e_plot],labels
    =['$E_{\mathscr{P}_r}G(p,k)$','$E_{\mathscr{P}(i)}G(p,k)$','$E_{\
   mathscr{P}(pm)}G(p,k)$','$E_{\mathscr{P}_s}G(p,k)$'])
plt.xlim(4,50)
plt.ylim(4,500)
plt.xlabel('p')
plt.ylabel(' $\mathscr{P}$-energy ')
plt.subplot(224)
in_e_plot, = plt.plot(t,in_e,'mo')
rb_e_plot, = plt.plot(t,rb_e,'c^')
sh_e_plot, = plt.plot(t,sh_e,'r+')
sp_e_plot, = plt.plot(t,sp_e,'y*')
plt.legend(handles = [rb_e_plot,in_e_plot,sp_e_plot,sh_e_plot],labels
    =['$E_{\mathscr{P}_r}G(p,k)$','$E_{\mathscr{P}(i)}G(p,k)$','$E_{\
   mathscr{P}(pm)}G(p,k)$','$E_{\mathscr{P}_s}G(p,k)$'])
plt.xlim(xmin,xmax)
plt.xlabel('p')
plt.ylabel(' $\mathscr{P}$-energy ')
plt.savefig("Plot4.png")
plt.show()
if __name__ == "__main__":
plot()
```

Figure 2: The comparison of $\mathscr{P}$-energy of $G(p,k)$ for different vertex partitions $\mathscr{P}$



Remark 2.102. Inference from Figure 2, leads us to the formulation of the following chain of inequalities for $G(p,k)$.

$$E_{\mathscr{P}_r}G(p,k) > E_{\mathscr{P}(i)}G(p,k) > E_{\mathscr{P}(pm)}G(p,k) > E_{\mathscr{P}_s}G(p,k).$$

2.5.2 A python program for determining different types of energies

The following program gives the exact values of energy, Laplacian energy, color energy, k-partition energy, $\mathscr{P}$-energy of a graph with the given number of vertices and vertex partitions.

```
   !pip install python-igraph
!pip install cairocffi
from igraph import *
import numpy as np
import numpy.linalg
dtype=np.float
from array import *
import networkx as nx
import matplotlib.pyplot as plt
n = int(input("Enter the number of vertices\n n="))
k = int(input("Total number of vertex partitions\n k="))
print("*****************************************************")
partition_order = []
vertex_partition_map = []
for i in range(0,k):
  partition_order.insert(i,0)
  print("i = " + str(i) + " vertex partition_order[i] = " + str(
    partition_order[i]))
for i in range(0,n):
    v=int(input("Enter the partition number of a vertex. \n(Specify
    partition number starting with 0.)\n Vertex " + str(i)+"=\n"))
    vertex_partition_map.insert(i,v)
    order = int(partition_order.pop(v))
    order = order+1
    partition_order.insert(v,order)
    str(order))
print("The orders of the vertex partitions corresponding to vertices
    0 to k are : \n", partition_order)
print("The vertex partition numbers of the vertices from 0 to k are
    :\n", vertex_partition_map)
print("*****************************************************")
# Create graph
g = Graph(directed=False)
G=nx.Graph()
#n = int(input("Enter the number of vertices="))
# Add 5 vertices
g.add_vertices(n)
G.add_node(n-1)
for i in range(len(g.vs)):
    g.vs[i]["id"]= i
```

```
    g.vs[i]["label"]= str(i)
option=[]
inputs=[]
for i in range (0,n):
  for j in range (0,n):
    if i <j:
          o = int(input("If vertex "+ str(i) + " and " + str(j) + "
    are adjacent type 1 and if not adjacent type 0 \n"))
          if (o==1):
             g.add_edges([(i,j)])
             G.add_edges_from([(i,j)])
print("*******************************************************")
print('edgelist:',G.edges())
nx.draw_circular(G)
plt.savefig('graph1.png')
plt.title("G")
plt.show()
print("*******************************************************")
print("Number of vertices in the graph:", g.vcount())
print("Number of edges in the graph", g.ecount())
print("Maximum degree in the graph:", g.maxdegree())
if i in range (0,n):
   print("The neighbors of the vertex " +str(i)+ " are :\n", g.
    neighbors(i, mode=ALL))
print("*******************************************************")
print("Adjacency matrix:\n", g.get_adjacency())
print("Laplacian matrix of a graph:\n",g.laplacian())
print("*******************************************************")
option=[]
adj_mat = g.get_adjacency()
adj = numpy.array(list(g.get_adjacency()))
amat=np.matrix(adj)
s = numpy.linalg.eigvals(adj)
print(s)
spec_abs1=np.abs(s)#gives absolute value
E1=np.sum(spec_abs1)
E2= format(E1, '.3f')
print("Energy is: \n",E2)
print("*******************************************************")
lap_mat =numpy.array(list(g.laplacian()))
lmat=np.matrix(lap_mat)
spec2=np.linalg.eigvals(lap_mat)
print("Laplacian Eigenvalues are:\n",spec2)
spec_abs2=np.abs(spec2)#gives absolute value
El2=np.sum(spec_abs2)
E3= format(El2, '.3f')
print("Laplacian Energy is: \n",E3)
```

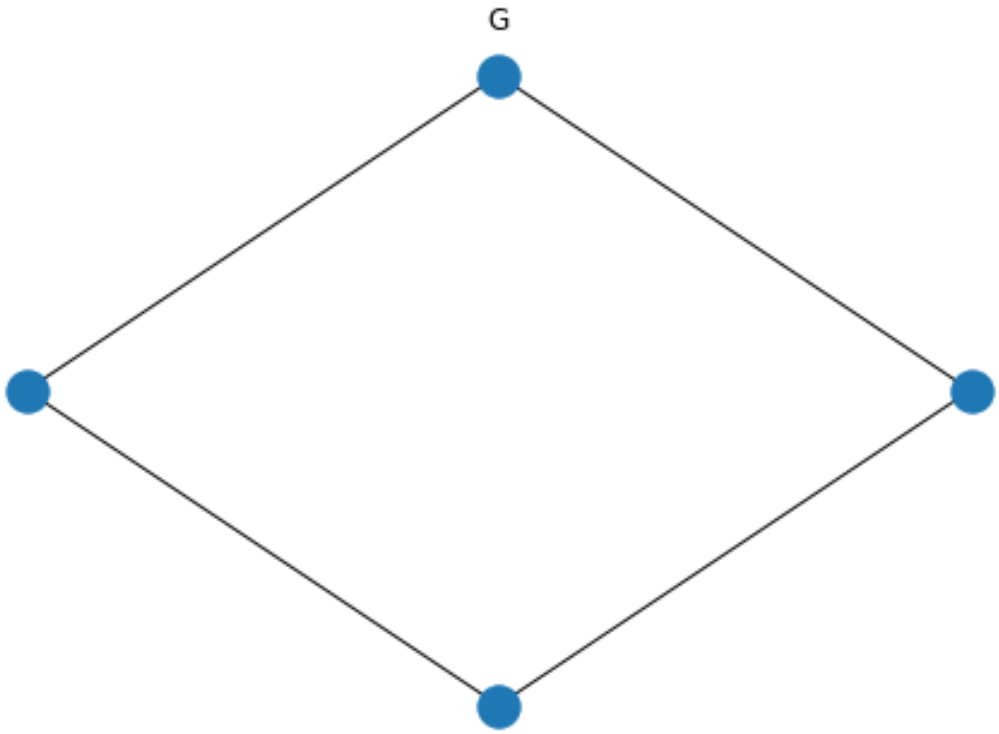
```
print("*****************************************************")
c=0
cmat = []
for i in range (0,n):
  cmat.append([])
for i in range (0,n):
  for j in range (0,n):
    cmat[i].append(j)
    cmat [i][j]=0
for i in range (0,n):
    for j in range (0,n):
      if (adj_mat[i][j]==1) and (vertex_partition_map[i] ==
    vertex_partition_map[j]):
          c=c+1
      else:
          if (i==j):
             cmat[i][j] = 0
          elif (vertex_partition_map[i] != vertex_partition_map[j]):
                  if (adj_mat[i][j]==1):
                      cmat[i][j] = 1
                  else:
                      cmat[i][j] = 0
          else:
                cmat[i][j] = -1
print(c)
if c>0:
  print("Color matrix can not be formed since partitions are not
    independent. ")
else:
    print("Color matrix:\n", cmat)
    clmat=np.matrix(cmat)
    np.set_printoptions(formatter={'float_kind':'{:f}'.format})
    spec5=np.linalg.eigvals(cmat)
    print("Color eigenvalues are:\n",spec5)
    spec_abs6=np.abs(spec5)#gives absolute value
    E6=np.sum(spec_abs6)
    E7= format(E6, '.3f')
    print("Color energy is:\n ",E7)
print("*****************************************************")
kmat = []
for i in range (0,n):
  kmat.append([])
for i in range (0,n):
  for j in range (0,n):
    kmat[i].append(j)
    kmat [i][j]=0
for i in range (0,n):
```

```
  for j in range (0,n):
    #print("i = " +str(i)+ " ; j = " +str(j))
    if(i==j):
      kmat[i][j] = 0
    elif (vertex_partition_map[i] != vertex_partition_map[j]):
      if(adj_mat[i][j]==1):
        kmat[i][j] = 1
      else:
        kmat[i][j] = 0
    else:
      if (adj_mat[i][j]==1):
        kmat[i][j] = 2
      else:
        kmat[i][j] = -1
print(str(k)+"-partition matrix:\n", kmat)
prmat=np.matrix(kmat)
np.set_printoptions(formatter={'float_kind':'{:f}'.format})#To
    convert the display of floats to decimals we use numpy.
    set_printoptions
spec4=np.linalg.eigvals(kmat)
print(str(k)+"-partition eigenvalues are:\n",spec4)
spec_abs4=np.abs(spec4)
#print(spec_abs)
E4=np.sum(spec_abs4)
E5= format(E4, '.3f')
print(str(k)+"-partition energy is:\n ",E5)
print("*****************************************************")
mat = []
for i in range (0,n):
  mat.append([])
for i in range (0,n):
  for j in range (0,n):
    mat[i].append(j)
    mat [i][j]=0
for i in range (0,n):
  for j in range (0,n):
    #print("i = " +str(i)+ " ; j = " +str(j))
    if(i==j):
      mat[i][j] = partition_order[vertex_partition_map[i]]
    elif (vertex_partition_map[i] != vertex_partition_map[j]):
      if(adj_mat[i][j]==1):
        mat[i][j] = 1
      else:
        mat[i][j] = 0
    else:
      if (adj_mat[i][j]==1):
        mat[i][j] = 2
```

```
      else:
        mat[i][j] = -1
print("P-matrix:\n", mat)
Matrix=np.matrix(mat)
#print("*****************************************************")
np.set_printoptions(formatter={'float_kind':'{:f}'.format})#To
    convert the display of floats to decimals we use numpy.
    set_printoptions
spec=np.linalg.eigvals(mat)
print("P-eigenvalues are:\n",spec)
spec_abs=np.abs(spec)#gives absolute value
E=np.sum(spec_abs)
E1= format(E, '.3f')
print("P-energy is:\n ",E1)
file= open('outfile.txt',"w")
file.write("Adjacency matrix is: \n")
for line in amat:
  np.savetxt(file, line, fmt='%.2f')
file.write("Energy is: \n")
file.write(E2)
file.write("Laplacian matrix is: \n")
for line in lmat:
  np.savetxt(file, line, fmt='%.2f')
file.write("Laplacian energy is: \n")
file.write(E3)
file.write("Color matrix is: \n")
for line in clmat:
  np.savetxt(file, line, fmt='%.2f')
file.write("Color energy is: \n")
file.write(E7)
file.write("Partition matrix is: \n")
for line in prmat:
  np.savetxt(file, line, fmt='%.2f')
file.write("Partition energy is: \n")
file.write(E5)
file.write("P-matrix is:\n ")
for line in Matrix:
  np.savetxt(file, line, fmt='%.2f')
file.write("P-energy is:\n ")
file.write( E1)
file.close()
```

The program gives the exact values for energy, Laplacian energy, color energy, partition energy and $\mathscr{P}$-energy of a graph. It is illustrated by the next example.

```
Requirement already satisfied: python-igraph in
/usr/local/lib/python3.6/dist-packages (0.8.3)
Requirement already satisfied: texttable>=1.6.2 in
/usr/local/lib/python3.6/dist-packages (from python-igraph) (1.6.3)
Requirement already satisfied: cairocffi in /usr/local/lib/python3.6/dist-
packages (1.2.0)
Requirement already satisfied: cffi>=1.1.0 in
/usr/local/lib/python3.6/dist-packages (from cairocffi) (1.14.4)
Requirement already satisfied: pycparser in /usr/local/lib/python3.6/dist-
packages (from cffi>=1.1.0->cairocffi) (2.20)
Enter the number of vertices
 n=4
Total number of vertex partitions
 k=2
**************************************************
i = 0 vertex partition_order[i] = 0
i = 1 vertex partition_order[i] = 0
Enter the partition number of a vertex.
(Specify partition number starting with 0.)
 Vertex 0=
0
Enter the partition number of a vertex.
(Specify partition number starting with 0.)
 Vertex 1=
1
Enter the partition number of a vertex.
(Specify partition number starting with 0.)
 Vertex 2=
1
Enter the partition number of a vertex.
(Specify partition number starting with 0.)
 Vertex 3=
0
The orders of the vertex partitions corresponding to vertices 0 to k are :
 [2, 2]
The vertex partition numbers of the vertices from 0 to k are :
 [0, 1, 1, 0]
**************************************************
If vertex 0 and 1 are adjacent type 1 and if not adjacent type 0
1
If vertex 0 and 2 are adjacent type 1 and if not adjacent type 0
0
If vertex 0 and 3 are adjacent type 1 and if not adjacent type 0
1
If vertex 1 and 2 are adjacent type 1 and if not adjacent type 0
1
If vertex 1 and 3 are adjacent type 1 and if not adjacent type 0
0
If vertex 2 and 3 are adjacent type 1 and if not adjacent type 0
1
**************************************************
edgelist: [(3, 0), (3, 2), (0, 1), (1, 2)]
```




```
******************************************************
Number of vertices in the graph: 4
Number of edges in the graph 4
Maximum degree in the graph: 2
The neighbors of the vertex 3 are :
 [0, 2]
******************************************************
Adjacency matrix:
 [[0, 1, 0, 1]
 [1, 0, 1, 0]
 [0, 1, 0, 1]
 [1, 0, 1, 0]]
Laplacian matrix of a graph:
 [[2, -1, 0, -1], [-1, 2, -1, 0], [0, -1, 2, -1], [-1, 0, -1, 2]]
******************************************************
[-2.000000 0.000000 2.000000 0.000000]
Energy is:
 4.000
******************************************************
Laplacian Eigenvalues are:
 [-0.000000 2.000000 4.000000 2.000000]
Laplacian Energy is:
 8.000
******************************************************
4
Color matrix can not be formed since partitions are not independent.
******************************************************
2-partition matrix:
 [[0, 1, 0, 2], [1, 0, 2, 0], [0, 2, 0, 1], [2, 0, 1, 0]]
2-partition eigenvalues are:
 [-3.000000 3.000000 -1.000000 1.000000]
2-partition energy is:
  8.000
```

```
**************************************************
P-matrix:
 [[2, 1, 0, 2], [1, 2, 2, 0], [0, 2, 2, 1], [2, 0, 1, 2]]
P-eigenvalues are:
 [-1.000000 5.000000 1.000000 3.000000]
P-energy is:
  10.000
```

3 Structure of the dissertation

In this dissertation, we study the concept color energy of a graph and its variations. We determine the exact values of color energy for few classes of graphs with respect to different coloring protocols. Then we generalized the concept and studied it in detail. Further, we adopt the programming approach towards the generalization of color energy. We organize the dissertation into six chapters.

In chapter 1, we present basic terminologies related to the energy and color energy of a graph. We mention a few graph and spectral theoretic results and definitions that lead us to a better understanding of the subsequent chapters. Moreover, we discuss the review of literature on energy and color energy in detail to throw some light on the research gaps which would be dealt with in the present study.

In chapter 2, we derive bounds for the color energy of a graph corresponding to various graph parameters such as $\chi(G), \gamma(G), \Delta(G)$, etc. Further, we examine the color energy with respect to proper colorings such as T-coloring, $L(h,k)$-coloring, radio coloring, and Hamiltonian coloring. Moreover, we compute the color energy of a few improper colorings such as open neighborhood coloring and improper c coloring.

In chapter 3, we propose a generalization of color energy, namely $\mathscr{P}$-energy of a graph. It is based on the $\mathscr{P}$-matrix. We obtain the bounds for $\mathscr{P}$-energy. Moreover, we determine the $\mathscr{P}$-energy of some standard classes of graphs like a complete graph, star graph, complete bipartite graph, etc.

In chapter 4, we investigate the $\mathscr{P}$-energy of generalized Petersen graphs $G(p,k)$ with different vertex partitions. We examine the relation between them and establish a chain of inequalities for $G(p,1)$.

In chapter 5, we obtain generalized expression for $\mathscr{P}$-energy of the join of graphs in terms of characteristic polynomials of the component graphs and the corresponding coronal. Further, using these expressions we derive the $\mathscr{P}$-energy of the join of graphs. Additionally, we investigate the explicit expressions for characteristic polynomial of the join of regular and complete bipartite graphs and obtain the corresponding $\mathscr{P}$-energy.

In chapter 6, we incorporate a programming approach in the present study. We develop a python program to determine the $\mathscr{P}$-energy of generalized Petersen graphs with the vertex partitions under consideration and compare their values. Moreover, we wrote a python program to find the energy, Laplacian energy, color energy, k-partition energy and $\mathscr{P}$-energy of a given graph.

References

[1] B. Acharya, S. Rao, P. Sumathi, and V. Swaminathan, "Energy of a set of vertices in a graph," *AKCE J. Graphs. Combin.*, vol. 4, no. 2, pp. 144–152, 2007.

[2] C. Adiga, E. Sampathkumar, and M. Sriraj, "Color energy of a unitary cayley graph," *Discuss. Math. Graph Theory*, vol. 34, no. 4, pp. 707–721, 2014.

[3] C. Adiga, E. Sampathkumar, M. Sriraj, and A. Shrikanth, "Color energy of a graph," *Proc. Jangjeon Math. Soc.*, vol. 16, no. 3, pp. 335–351, 2013.

[4] N. Alon, G. Fertin, A. L. Liestman, T. C. Shermer, and L. Stacho, "Factor d-domatic colorings of graphs," *Discrete Mathematics*, vol. 262, no. 1-3, pp. 17–25, 2003.

[5] D. Archdeacon, M. Debowsky, J. Dinitz, and H. Gavlas, "Cycle systems in the complete bipartite graph minus a one-factor," *Discrete mathematics*, vol. 284, no. 1-3, pp. 37–43, 2004.

[6] S. K. R. Avula, P. S. K. Reddy, and K. Prakasha, "Minimum equitable dominating partition energy of a graph," *Journal of Informatics and Mathematical Sciences*, vol. 10, no. 1-2, pp. 13–22, 2018.

[7] A. Benjamin, G. Chartrand, and P. Zhang, *The fascinating world of graph theory*. Princeton University Press, 2017.

[8] K. S. Betagiri, "The reduced color energy of graphs," *J. Comp. and Math. sci.*, vol. 7, no. 3, pp. 13–20, Jan. 2016.

[9] P. B. Bhat and S. D'souza, "Color laplacian energy of a graph," *Proc. Jangjeon Math. Soc.*, vol. 18, no. 3, pp. 321–330, 2015.

[10] ——, "Color signless laplacian color energy of graphs," *J. Comp. and Math. sci.*, vol. 14, no. 2, pp. 142–148, Aug. 2017.

[11] N. Biggs, *Algebraic Graph Thoery*. Cambridge Univ. Press, Cambridge, UK, 1993.

[12] C. Bujtás, E. Sampathkumar, Z. Tuza, L. Pushpalatha, and R. Vasundhara, "Improper c-colorings of graphs," *Discrete applied mathematics*, vol. 159, no. 4, pp. 174–186, 2011.

[13] G. Chartrand and P. Zhang, *Chromatic Graph Theory*. New York: Chapman and Hall/CRC, 2009.

[14] G. Chartrand, H. Hevia, and R. J. Wilson, "The ubiquitous petersen graph," *The Julius Petersen graph theory centennial*, vol. 6, p. 303, 1992.

[15] E. J. Cockayne, R. Dawes, and S. T. Hedetniemi, "Total domination in graphs," *Networks*, vol. 10, no. 3, pp. 211–219, 1980.

[16] E. J. Cockayne and S. T. Hedetniemi, "Towards a theory of domination in graphs," *Networks*, vol. 7, no. 3, pp. 247–261, 1977.

[17] V. Consonni and R. Todeschini, "New spectral indices for molecule description," *Match*, vol. 1, p. 2, 2008.

[18] H. S. Coxeter, "Self-dual configurations and regular graphs," *Bulletin of the American Mathematical Society*, vol. 56, no. 5, pp. 413–455, 1950.

[19] S.-Y. Cui and G.-X. Tian, "The spectrum and the signless laplacian spectrum of coronae," *Linear algebra and its applications*, vol. 437, no. 7, pp. 1692–1703, 2012.

[20] D. M. Cvetković, M. Doob, and H. Sachs, *Spectra of Graphs- Theory and Application*. Academic Press, New York, 1980.

[21] P. J. Davis, *Circulant matrices*. Wiley, New York, 1979.

[22] J. B. Diaz, F. T. Metcalf *et al.*, "Stronger forms of a class of inequalities of g. pólya-g. szegö, and lv kantorovich," *Bulletin of the American Mathematical Society*, vol. 69, no. 3, pp. 415–418, 1963.

[23] E. Estrada and M. Benzi, "What is the meaning of the graph energy after all?" *Discrete Applied Mathematics*, vol. 230, pp. 71–77, 2017.

[24] M. Farber, G. Hahn, P. Hell, and D. Miller, "Concerning the achromatic number of graphs," *Journal of Combinatorial Theory, Series B*, vol. 40, no. 1, pp. 21–39, 1986.

[25] K. Geetha, K. Meera, N. Narahari, and B. Sooryanarayana, "Open neighborhood coloring of graphs," *Int. J. Contemp. Math. Sciences*, vol. 8, no. 14, pp. 675–686, 2013.

[26] M. Girvan and M. E. Newman, "Community structure in social and biological networks," *Proceedings of the national academy of sciences*, vol. 99, no. 12, pp. 7821–7826, 2002.

[27] R. Guimera and L. A. N. Amaral, "Functional cartography of complex metabolic networks," *nature*, vol. 433, no. 7028, pp. 895–900, 2005.

[28] I. Gutman, "The energy of a graph," *Ber. Math. Stat. Sekt. Forschungsz. Graz*, vol. 103, pp. 1–22, 1978.

[29] ——, "The energy of a graph: old and new results," in *Algebraic combinatorics and applications*. Springer, 2001, pp. 196–211.

[30] I. Gutman and B. Furtula, "The total π-electron energy saga," *Croatica Chemica Acta*, vol. 90, no. 3, pp. 1–10, 2017.

[31] I. Gutman, X. Li, and J. Zhang, *Graph Energy*. Springer, New York, 2012.

[32] I. Gutman and B. Zhou, "Laplacian energy of a graph," *Linear Algebra and its applications*, vol. 414, no. 1, pp. 29–37, 2006.

[33] S. Hedetniemi, P. Slater, and T. W. Haynes, *Fundamentals of Domination in Graphs*. Marcel De. Inc., New York, 1998.

[34] B. Horvat, T. Pisanski, and A. Žitnik, "Isomorphism checking of i-graphs," *Graphs and Combinatorics*, vol. 28, no. 6, pp. 823–830, 2012.

[35] B. Horvat and T. Pisanski, "Products of unit distance graphs," *Discrete Mathematics*, vol. 310, no. 12, pp. 1783–1792, 2010, algebraic and Topological Graph Theory. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0012365X09005949

[36] G. Indulal, I. Gutmanb, and A. Vijayakumar, "On distance energy of graphs," *MATCH Comm. Math. Comp. Chem.*, vol. 60, pp. 461–472, 2008.

[37] P. B. Joshi and M. Joseph, "Further results on color energy of graphs," *Acta Univ. Sapient., Info.*, vol. 9, no. 2, pp. 191–131, 2017.

[38] ——, "On new bounds for color energy of graphs," *Int. J. Pure and appl. math.*, vol. 117, no. 11, pp. 25–33, Dec. 2017.

[39] ——, "$\mathscr{P}$-energy of graphs," *Acta Univ. Sapientiae, Informatica*, vol. 12, no. 1, pp. 137–157, 2020.

[40] ——, "New results on $\mathscr{P}$-energy of join of graphs," *Malaya Journal of Matematik*, vol. 8, no. 4, pp. 2092–2096, 2020.

[41] ——, "On $\mathscr{P}$-energy of join of graphs," *Malaya Journal of Matematik*, vol. 8, no. 4, pp. 2082–2087, 2020.

[42] M. R. Kanna, R. P. Kumar, and R. Jagadeesh, "Minimum covering color energy of a graph," *Int. J. Math. Anal.*, vol. 9, no. 8, pp. 351–364, 2015.

[43] X. Liu and Z. Zhang, "Spectra of subdivision-vertex join and subdivision-edge join of two graphs," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 1, pp. 15–31, 2019.

[44] A. Lupaş, "A remark on the schweitzer and kantorovich inequalities," *Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika*, no. 381/409, pp. 13–15, 1972.

[45] M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*. Courier Corporation, 1992, vol. 14.

[46] C. McLeman and E. McNicholas, "Spectra of coronae," *Linear algebra and its applications*, vol. 435, no. 5, pp. 998–1007, 2011.

[47] C. D. Meyer, *Matrix analysis and applied linear algebra*. Siam, 2000, vol. 71.

[48] T. Moscibroda and R. Wattenhofer, "Maximizing the lifetime of dominating sets," in *19th IEEE International Parallel and Distributed Processing Symposium*. IEEE, 2005, pp. 8–pp.

[49] K. Murthy and B. Dhananjayamurthy, "On domatic energy of a graph," *Int.l J.l Math. Appl.*, vol. 5, no. 4, pp. 699–709, 2017.

[50] M. Naor and L. Stockmeyer, "What can be computed locally?" *SIAM Journal on Computing*, vol. 24, no. 6, pp. 1259–1277, 1995.

[51] N. Ozeki, "On the estimation of the inequalities by the maximum, or minimum values," *J. Colledge Arts Ci*, vol. 5, pp. 199–203, 1968.

[52] T. Pisanski and M. Krnc, "Generalized petersen graphs and kronecker covers," *Discrete Mathematics & Theoretical Computer Science*, vol. 21, 2019.

[53] G. Pólya and G. Szegö, "Inequalities," in *Problems and Theorems in Analysis*. Springer, 1972, pp. 62–75.

[54] J. Rada, "The linear chain as an extremal value of vdb topological indices of polyomino chains," *Appl. Math. Sci*, vol. 8, pp. 5133–5143, 2014.

[55] S. Radenković and I. Gutman, "Total π-electron energy and laplacian energy: How far the analogy goes?" *Journal of the Serbian Chemical Society*, vol. 72, no. 12, pp. 1343–1350, 2007.

[56] M. R. Rajesh Kanna and R. Jagadeesh, "Distinct energies of paracetamol," *Int. J. Math. And Appl.*, vol. 6, pp. 781–789, 2018.

[57] P. S. K. Reddy, K. Prakasha, and S. Vijay, "Additive color laplacian energy of a graph," *Journal of Research in Science Technology Engineering and Management*, vol. 2, no. 4, pp. 106–109, 2016.

[58] S. K. Richard Hammack, Wilfried Imrich, *Handbook of Product Graphs, Second Edition*, 2nd ed., ser. Discrete Mathematics and Its Applications. CRC Press, 2011. [Online]. Available: http://gen.lib.rus.ec/book/index.php?md5=A491E3CB93630CC8761EF0CF96770D94

[59] E. Sampathkumar and L. Pushpalatha, "Complement of a graph: A generalization," *Graphs Comb.*, vol. 14, no. 4, pp. 377–392, 1998.

[60] E. Sampathkumar, L. Pushpalatha, C. Venkatachalam, and P. Bhat, "Generalized complements of a graph," *Indian J. of Pure and App. Math.*, pp. 625–639, 1998.

[61] E. Sampathkumar, S. V. Roopa, K. A. Vidya, and M. A. Sriraj, "Partition energy of a graph," *Proc. Jangjeon Math. Soc.*, vol. 18, no. 4, pp. 473–493, 2015.

[62] E. Sampathkumar, S. Roopa, K. Vidya, and M. Sriraj, "Partition laplacian energy of a graph," *Palestine Journal of Mathematics*, vol. 8, no. 1, 2019.

[63] E. Sampathkumar and M. A. Sriraj, "Vertex labeled/colored graphs, matrices and signed graphs," *J. Comb. Info. System Sci.*, vol. 38, no. 1-4, pp. 113–120, 2013.

[64] V. S. Shigehalli and K. S. Betageri, "Color laplacian energy of graphs," *J. Comp. and Math. sci.*, vol. 6, no. 9, pp. 485–595, Sep. 2015.

[65] V. Shigehalli and K. S. BETAGERI, "A note on color energy and color laplacian energy of graphs," *International Journal of Mathematical Archive EISSN 2229-5046*, vol. 7, no. 10, 2016.

[66] F. Z. e. Simo Puntanen, George P. H. Styan (auth.), *The Schur Complement and Its Applications*, 1st ed., ser. Numerical Methods and Algorithms 4. Springer US, 2005. [Online]. Available: http://gen.lib.rus.ec/book/index.php?md5=D930969FB8A4FCA329669E2A598F0E21

[67] M. Sriraj, "Bounds for the largest color eigenvalue and the color energy," *Mathematical Combinatorics*, vol. 1, pp. 127–134, 2017.

[68] M. E. Watkins, "A theorem on tait colorings with an application to the generalized petersen graphs," *Journal of Combinatorial Theory*, vol. 6, no. 2, pp. 152–164, 1969.

[69] E. W. Weisstein, "Generalized petersen graph," *https://mathworld. wolfram. com/*, 2002.

[70] D. B. West, *Introduction to Graph Theory*. Pearson, New Jersey, 2001.

[71] H. Wolkowicz and G. P. Styan, "Bounds for eigenvalues using traces," *Linear algebra appl.*, vol. 29, pp. 471–506, 1980.

[72] B. Zelinka, "On *k*-ply domatic numbers of graphs," *Mathematica Slovaca*, vol. 34, no. 3, pp. 313–318, 1984.

[73] ——, "Domination in generalized petersen graphs," *Czechoslovak Mathematical Journal*, vol. 52, no. 1, pp. 11–16, 2002.

[74] M. Zhang, J. Deng, C. V. Fang, X. Zhang, and L. J. Lu, "Molecular network analysis and applications," *Knowledge-based bioinformatics: from analysis to interpretation*, pp. 253–287, 2010.

[75] A. Zitnik, B. Horvat, and T. Pisanski, "All generalized petersen graphs are unit-distance graphs," *J. Korean Math. Soc*, vol. 49, no. 3, pp. 475–491, 2012.

LIST OF PUBLICATIONS

1. P. B. Joshi and M. Joseph, "On color energy of few classes of bipartite graphs and corresponding color complements," Mathematical Journal of Interdisciplinary Sciences, vol. 8, no. 1, pp. 7-14, 2019.

2. P. B. Joshi and M. Joseph, "$\mathscr{P}$-energy of graphs, " Acta Univ. Sapientiae, Informatica, vol. 12, no. 1, pp. 133–153, 2020.

3. P. B. Joshi and M. Joseph, "On $\mathscr{P}$-energy of join of graphs " Malaya Journal of Matematik, Vol. 8, no. 4, pp. 2082–2087, 2020.

4. P. B. Joshi and M. Joseph, "New results on $\mathscr{P}$-energy of join of graphs " Malaya Journal of Matematik, Vol. 8, no. 4, pp. 2092–2096, 2020.

5. P. B. Joshi and M. Joseph, " $\mathscr{P}$-energy of generalized Petersen graphs", Commun. Comb. Optim., [Online]. Doi: https://doi.org/ 10.22049/CCO.2021.27102.1197

LIST OF PAPER PRESENTATIONS

1. A paper entitled "New Results on Color Energy of Graphs" was presented in the "National Conference on Emerging Trends in Mathematics and Technology" organized by Hindustan Institute of Technology and Science (Deemed to be University), Chennai during $22^{nd} - 23^{rd}$, March 2018.

2. A paper entitled "On Color Energy of a Graph and its Color Complement" was presented in the "National Conference on Recent Trends in Mathematics" organized by Dept. of Maths., GITAM (Deemed to be University), Bengaluru during $21^{st} - 22^{nd}$, December 2018.

3. A paper entitled "The Domatic Color Energy of a Graph" was presented in the "International Conference on Emerging Trends in Graph Theory" organized by Dept. of Maths., CHRIST(Deemed to be University), Bengaluru during $27^{th} - 28^{th}$, February 2019.

4. A paper entitled "The Domatic Color Energy of a Graph" was presented in the "International Conference on Number Theory and Graph Theory" organized by Dept. of Studies in Maths., University of Mysore, Mysore during $27^{th} - 29^{th}$, June 2019.

5. A paper entitled “On $\mathscr{P}$-energy of join of graphs" was presented in the International Conference on Graph Connections organized by Department of Mathematics and statistics, Bishop Chulaparambil Memorial College, Kottayam during 6-8 August 2020.